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Journal of Geometry and Physics 57 (2007) 2098–2113
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Homogeneous quaternionic Kähler structures on 12-dimensional Alekseevsky spaces[☆]

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Received 12 December 2006; received in revised form 25 April 2007; accepted 22 May 2007
Available online 8 June 2007

Abstract

For each Alekseevsky space of dimension 12, its description as a homogeneous Riemannian space, and the homogeneous quaternionic Kähler structures that it admits through Witte's refined Langlands decomposition, are given.

MSC: primary 53C26; secondary 53C30; 53C35

Keywords: Alekseevsky symmetric spaces; Homogeneous Riemannian structures; Homogeneous quaternionic Kähler structures; Parabolic subgroups; Refined Langlands decomposition

1. Introduction and preliminaries

1.1. Introduction

Quaternion–Kähler symmetric spaces were classified by Wolf [14] and homogeneous quaternionic Kähler structures were classified by Fino [8] (cf. [5]). The study of the types of homogeneous quaternionic Kähler structures appearing on negative quaternion–Kähler symmetric spaces arises as a natural question. These spaces are Alekseevskian (Alekseevsky [1], Cortés [7]). That study was started in [5] for the quaternionic hyperbolic space and in a previous paper by the authors [4] for the case of dimension 8.

In the present paper we obtain, for each Alekseevsky space of dimension 12, the homogeneous quaternionic Kähler structures that it admits through Witte's refined Langlands decomposition [13]. We first find the connected closed cocompact subgroups acting transitively by isometries on each one of the 12-dimensional Alekseevsky spaces. We further obtain the type of homogeneous quaternionic Kähler structures on each of these spaces, in terms of the five primitive classes $\mathcal{QK}_1, \ldots, \mathcal{QK}_5$ in Fino's classification. Theorem 5 sums up some of the results throughout the paper.

☆ Partially supported by DGICYT, Spain, under Grant MTM2005–00173.
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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.05.007

On the other hand, it is well known that Alekseevsky spaces play an important role in $d = 4$, $N = 2$ supergravity, as target spaces of the hypermultiplet sector of sigma models (see among others Cecotti [6], de Wit and van Proeyen [12]). To quote but an example of the interest in physics of the spaces studied in [4] and in the present paper, we recall that they are spaces originated by either the c-map or by the $c \circ r$ map, as follows. The real projective spaces are the origin, under the $c \circ r$ map, of Alekseevsky spaces of rank 3, that is, the spaces $\mathcal{T}(p)$, with $p \geqslant 0$ (see [1,12,7]). The only symmetric space in the series is $\mathcal{T}(0) \cong SO_0(4,3)/S(O(4) \times O(3))$, which comes from $(c \circ r)(SO(1,1)) = c((SU(1,1)/U(1))^2)$. The minimal couplings of vector multiplets in $d = 4$, $N = 2$ supergravity, the complex hyperbolic spaces $\mathbb{C}\mathbb{H}(n)$, originate under the c-map, the infinite series of rank 2 quaternion–Kähler symmetric spaces $SU(n,2)/S(U(n) \times U(2))$. In this paper, the case $n = 3$ is considered. The rank 1 quaternionic hyperbolic space $\mathbb{H}\mathrm{H}(n)$ comes by the c-map from pure $d = 4$ supergravity, i.e., from the empty special Kähler space.

1.2. Homogeneous quaternionic Kähler structures

Let (M, g) be a connected, simply connected, and complete Riemannian manifold. Ambrose and Singer [3] gave a characterization for (M, g) to be homogeneous in terms of a $(1, 2)$ tensor field S, usually called a homogeneous Riemannian structure. Let ∇ be the Levi-Civita connection of g and R its curvature tensor. Then the manifold is homogeneous if and only if the Ambrose–Singer equations $\widetilde{\nabla} g = 0$, $\widetilde{\nabla} R = 0$, $\widetilde{\nabla} S = 0$, where $\widetilde{\nabla} = \nabla - S$, are satisfied.

Suppose now that (M, g, υ) is a quaternion–Kähler manifold, where υ denotes the distinguished rank 3 subbundle of the bundle of $(1, 1)$ tensor fields on M. Such a manifold is a *homogeneous quaternion–Kähler space* if it admits a transitive group of isometries [2]. We have (cf. [3,8]) as a corollary to Kiričenko's theorem [11], that a connected, simply connected, and complete quaternionic Kähler manifold (M, g, υ) is homogeneous if and only if there exists a tensor field S of type $(1, 2)$ on M satisfying $\widetilde{\nabla} g = 0$, $\widetilde{\nabla} R = 0$, $\widetilde{\nabla} S = 0$, $\widetilde{\nabla} \Omega = 0$, where $\widetilde{\nabla} = \nabla - S$ and Ω is the canonical 4-form of (M, g, υ). Then S is said to be a *homogeneous quaternionic Kähler structure* on M. Defining $S_{XYZ} = g(S_X Y, Z)$, the condition $\widetilde{\nabla} \Omega = 0$ can be replaced by the equation

$$S_{XJ_1YJ_1Z} - S_{XYZ} = \theta^3(X) g(J_2Y, J_1Z) - \theta^2(X) g(J_3Y, J_1Z), \tag{1.1}$$

and the equations obtained by a cyclic permutation of the indices $1, 2, 3$, for certain differential 1-forms θ^a, $a = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ is a local basis of υ satisfying the conditions $J_a^2 = -I$, $J_a J_b = -J_b J_a = J_c$, for each cyclic permutation (a, b, c) of $(1, 2, 3)$. Let $(V, \langle,\rangle, J_1, J_2, J_3)$ be a quaternion–Hermitian real vector space, i.e., a $4n$-dimensional real vector space endowed with an inner product $\langle,\rangle$ and operators J_1, J_2, J_3, satisfying $J_1^2 = J_2^2 = J_3^2 = -I$, $J_1 J_2 = -J_2 J_1 = J_3$ and the two other similar relations, and $\langle J_a X, J_a Y\rangle = \langle X, Y\rangle$, $a = 1, 2, 3$. Such a space V is the model for the tangent space at any point of a quaternion–Kähler manifold. Consider the space of tensors $\mathcal{T}(V) = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY}\}$, and its vector subspace $\mathcal{V}$ of tensors satisfying Eq. (1.1) with $\langle,\rangle, \theta^a \in V^*$. Then $\mathcal{V} = \check{\mathcal{V}} + \hat{\mathcal{V}}$, where $\check{\mathcal{V}} = \{\Theta \in \otimes^3 V^* : \Theta_{XYZ} = \sum_{a=1}^3 \theta^a(X) \langle J_a Y, Z\rangle, \theta^a \in V^*\}$, and $\hat{\mathcal{V}} = \{T \in \otimes^3 V^* : T_{XYZ} = -T_{XZY}, T_{XJ_aYJ_aZ} = T_{XYZ}, a = 1, 2, 3\}$. This decomposition of $\mathcal{V}$ is orthogonal with respect to the scalar product (,) defined by $(S, S') = \sum_{r,s,t=1}^{4n} S_{e_r e_s e_t} S'_{e_r e_s e_t}$, where $\{e_r\}_{r=1,\ldots,4n}$ is an orthonormal basis of V. The spaces $\check{\mathcal{V}}$ and $\hat{\mathcal{V}}$ decompose respectively into two and three subspaces, giving an orthogonal sum of five subspaces which are invariant and irreducible under the action of $Sp(n)Sp(1)$, as proved by using the next theorem. Let E denote the standard representation of $Sp(n)$ on $\mathbb{C}^{2n}$; $S^3 E$ the 3-symmetric product of E; K the irreducible $Sp(n)$-module of highest weight $(2, 1, 0, \ldots, 0)$; H the standard representation of $Sp(1)$ on $\mathbb{C}^2$; and $S^3 H$ the four-dimensional symmetric product of H. Denoting real representations with brackets and with the usual notation, we have

Theorem 1 (*Fino [8]*). *The space $[EH] \otimes (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n))$ of homogeneous quaternionic Kähler structures splits into invariant and irreducible subspaces under the action of $Sp(n)Sp(1)$ as $[EH] + [ES^3H] + [EH] + [S^3EH] + [KH]$.*

Denoting by $\mathcal{QK}_i$ the ith summand in Fino's classification, we have

Theorem 2 (*[5]*). *If $n \geqslant 2$, then $\mathcal{V}$ decomposes into the direct sum of the following subspaces invariant and irreducible under the action of $Sp(n)Sp(1)$:*

$$\mathcal{QK}_1 = \left\{ \Theta \in \check{\mathcal{V}} : \Theta_{XYZ} = \sum_{a=1}^{3} \theta(J_a X) \langle J_a Y, Z \rangle, \theta \in V^* \right\},$$

$$\mathcal{QK}_2 = \left\{ \Theta \in \check{\mathcal{V}} : \Theta_{XYZ} = \sum_{a=1}^{3} \theta^a(X) \langle J_a Y, Z \rangle, \ \sum_{a=1}^{3} \theta^a \circ J_a = 0, \ \theta^a \in V^* \right\},$$

$$\mathcal{QK}_3 = \left\{ T \in \hat{\mathcal{V}} : T_{XYZ} = \langle X, Y \rangle \vartheta(Z) - \langle X, Z \rangle \vartheta(Y) + \sum_{a=1}^{3} (\langle X, J_a Y \rangle \vartheta(J_a Z) - \langle X, J_a Z \rangle \vartheta(J_a Y)), \vartheta \in V^* \right\},$$

$$\mathcal{QK}_4 = \left\{ T \in \hat{\mathcal{V}} : T_{XYZ} = \frac{1}{2} \left(T_{YZX} - T_{ZXY} + \sum_{a=1}^{3} (T_{J_a Y J_a Z X} - T_{J_a Z J_a Y X}) \right), c_{12}(T) = 0 \right\},$$

$$\mathcal{QK}_5 = \left\{ T \in \hat{\mathcal{V}} : T_{XYZ} = -\frac{1}{4} \left(T_{YZX} - T_{ZYX} + \sum_{a=1}^{3} (T_{J_a Y J_a Z X} - T_{J_a Z J_a Y X}) \right) \right\}.$$

Denoting the sum of classes $\mathcal{QK}_i + \mathcal{QK}_j$ by $\mathcal{QK}_{ij}$ and so on, we have $\check{\mathcal{V}} = \mathcal{QK}_{12}$, $\hat{\mathcal{V}} = \mathcal{QK}_{345}$.

1.3. Cocompact subgroups acting transitively

Gordon and Wilson gave in [9] a theorem of characterization of the isometry groups acting transitively on noncompact Riemannian symmetric spaces. We proved in [4] Theorem 4 below, which is related to Witte's Theorem 3 and suffices for our purposes. The set-up is as follows. Let (M, g) be a connected noncompact Riemannian manifold and G its full connected isometry group. If K is the isotropy group at any fixed point $o \in M$, then $M = G/K$. We look for the subgroups $\hat{G}$ of G acting transitively by isometries on M. Defining $K_{\hat{G}} = \hat{G} \cap K$, one must have $M \equiv \hat{G}/K_{\hat{G}}$, and thus all the descriptions of (M, g) as a homogeneous Riemannian space. If $\hat{G}$ is a closed subgroup of G which acts transitively on M, then it is cocompact, that is, $G/\hat{G}$ is compact.

The structure of the nondiscrete cocompact subgroups of a connected semisimple Lie group with finite center was given by Witte in [13] (cf. [10]), as follows. Let $\mathfrak{g}$ be the Lie algebra of such a Lie group, $\mathfrak{a}$ a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$, Σ the set of roots of $(\mathfrak{g}, \mathfrak{a})$, and $\mathfrak{g} = \mathfrak{g}_0 + \sum_{f \in \Sigma} \mathfrak{g}_f$ the restricted-root space decomposition, with $\mathfrak{g}_0 = \mathfrak{a} + Z_{\mathfrak{k}}(\mathfrak{a})$, where $Z_{\mathfrak{k}}(\mathfrak{a})$ stands for the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Write Σ^+ for the set of positive roots with respect to a certain notion of positivity for $\mathfrak{a}^*$, and let Π be the set of simple restricted roots. For each subset Ψ of Π, let $[\Psi]$ be the set of restricted roots that are linear combinations of elements of Ψ. Then, the standard parabolic subgroup P_Ψ^0 is defined as the connected subgroup of G having Lie algebra $\mathfrak{p}_\Psi = \mathfrak{g}_0 + \sum_{f \in \Sigma^+ \cup [\Psi]} \mathfrak{g}_f = \mathfrak{l}' + \mathfrak{n}_\Psi$, where $\mathfrak{l}' = \mathfrak{g}_0 + \sum_{f \in [\Psi]} \mathfrak{g}_f = \mathfrak{l}'_\Psi + \mathfrak{e}'_\Psi + \mathfrak{a}_\Psi$, with $\mathfrak{l}'_\Psi$ semisimple with noncompact summands, $\mathfrak{e}'_\Psi$ compact reductive, $\mathfrak{a}_\Psi$ the noncompact part of the center of $\mathfrak{l}'$, and $\mathfrak{n}_\Psi = \sum_{f \in \Sigma^+ \setminus [\Psi]} \mathfrak{g}_f$ nilpotent. On the Lie group level one has ([13]) the *refined Langlands decomposition* $P_\Psi^0 = L'_\Psi E'_\Psi A_\Psi N_\Psi$ and

Theorem 3 (*Witte [13]*). *Let L be a connected normal subgroup of L'_Ψ and E a connected closed subgroup of $E'_\Psi A_\Psi$. Then there is a closed cocompact subgroup $\hat{G}$ of G contained in P_Ψ with identity component $\hat{G}^0 = LEN_\Psi$. Moreover, every closed cocompact subgroup of G arises in this way.*

Furthermore, we proved in [4].

Theorem 4. *Let G be a connected semisimple Lie group with finite center and $G = KAN$ an Iwasawa decomposition. A connected closed cocompact subgroup $\hat{G} = LEN_\Psi$ of G acts transitively on $M = G/K$ if and only if:* (a) *The projections of the Lie algebra $\mathfrak{l} \subset \mathfrak{l}' = \mathfrak{g}_0 + \sum_{f \in [\Psi]} \mathfrak{g}_f$ of L to $\sum_{f \in \Sigma^+ \cap [\Psi]} \mathfrak{g}_f$ and to $\mathfrak{a}_\Psi^\perp$ are surjective, $\mathfrak{a}_\Psi^\perp$ being* *the orthogonal complement to* $\mathfrak{a}_\Psi$ *in* $\mathfrak{a} \subset \mathfrak{g}_0 = \mathfrak{a}_\Psi + \mathfrak{a}_\Psi^\perp + Z_{\mathfrak{k}}(\mathfrak{a})$. (b) *The projection of the Lie algebra* $\mathfrak{e} \subset \mathfrak{e}'_\Psi + \mathfrak{a}_\Psi$ *of* E *to* $\mathfrak{a}_\Psi$ *is surjective.*

1.4. Homogeneous Riemannian structures on symmetric Alekseevsky spaces

The Alekseevsky spaces [1,7] are the nonflat quaternion–Kähler spaces which admit a simply transitive real solvable group of isometries. The noncompact duals of the Wolf spaces are Alekseevsky spaces. Moreover, the Alekseevsky spaces of dimension smaller than 16 are symmetric.

Let M be a connected noncompact quaternion–Kähler symmetric space. Then $M = G/K$, where G is the connected component of the identity of the isometry group of M and K is the isotropy subgroup of G at a point $o \in M$. We consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, and the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{a} \subset \mathfrak{p}$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$, and $\mathfrak{n}$ is a nilpotent subalgebra. Let A and N be the connected abelian and nilpotent Lie subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group AN acts simply transitively on M. Suppose now that $\hat{G}$ is a connected closed Lie subgroup of G which acts transitively on M. The isotropy group of this action at $o = K \in M$ is $H = K_{\hat{G}} = \hat{G} \cap K$. Then $M = G/K$ has also the description $M \equiv \hat{G}/H$, and $o \equiv H \in \hat{G}/H$. Consider a reductive decomposition of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$, that is, a vector space direct sum $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$. Since $\hat{G}$ is connected and M is simply connected then H is connected, and the condition $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. We have the isomorphisms of vector spaces

$$\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k} \cong \hat{\mathfrak{g}}/\mathfrak{h} \cong \mathfrak{m} \cong T_o(M) \cong \mathfrak{a} + \mathfrak{n}, \tag{1.2}$$

with $\xi\colon \mathfrak{p} \overset{\cong}{\to} \mathfrak{m}$, $\mu\colon \mathfrak{m} \overset{\cong}{\to} T_o(M)$, and $\zeta\colon T_o(M) \overset{\cong}{\to} \mathfrak{a} + \mathfrak{n}$, given by $\xi^{-1}(Z) = Z_{\mathfrak{p}}$ and $\mu(Z) = Z^*_o$ for $Z \in \mathfrak{m}$, and $\zeta^{-1}(X) = X^*_o$ for $X \in \mathfrak{a} + \mathfrak{n}$, where, for each $X \in \mathfrak{g}$, X^* denotes the vector field on M generated by the one-parameter subgroup $\{\exp tX\}$ of G acting on M. The scalar product induced in $\mathfrak{a} + \mathfrak{n}$ by the isomorphisms in (1.2) and a positive multiple of $B_{|\mathfrak{p}\times\mathfrak{p}}$, where B is the Killing form of $\mathfrak{g}$, define a left-invariant Riemannian metric on AN such that AN is isometric to M.

The reductive decomposition $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ defines the homogeneous Riemannian structure $S = \nabla - \widetilde{\nabla}$, where $\widetilde{\nabla}$ is the canonical connection of $M \equiv \hat{G}/H$ with respect to $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$, and it is $\hat{G}$-invariant and uniquely determined by $(\widetilde{\nabla}_{X^*} Y^*)_o = -[X, Y]^*_o$, for $X, Y \in \mathfrak{m}$. Now, if $X \in \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we write $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$, $(X_{\mathfrak{k}} \in \mathfrak{k}$, $X_{\mathfrak{p}} \in \mathfrak{p})$, and if $X, Y \in \mathfrak{m}$, then $(X_{\mathfrak{k}})^*_o = 0$ and $(\nabla(X_{\mathfrak{p}})^*)_o = 0$, hence $S_{X^*_o} Y^*_o = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}]^*_o$. Thus, for each $X, Y \in \mathfrak{a} + \mathfrak{n}$, we have

$$S_{X^*_o} Y^*_o = S_{\xi(X_{\mathfrak{p}})^*_o} \xi(Y_{\mathfrak{p}})^*_o = [(\xi(X_{\mathfrak{p}}))_{\mathfrak{k}}, Y_{\mathfrak{p}}]^*_o. \tag{1.3}$$

The quaternionic structure on M is defined by a three-dimensional ideal $\mathfrak{u} = \langle E_1, E_2, E_3 \rangle \cong \mathfrak{sp}(1)$ of $\mathfrak{k}$, where $[E_1, E_2] = 2E_3$, $[E_2, E_3] = 2E_1$, $[E_3, E_1] = 2E_2$. The endomorphisms ad_{E_i} of $\mathfrak{p}$, $(1 \leqslant i \leqslant 3)$, and the isomorphisms in (1.2) define the complex structures $J_i \in \mathrm{End}(\mathfrak{a} + \mathfrak{n})$, $(1 \leqslant i \leqslant 3)$, which make $(\mathfrak{a} + \mathfrak{n}, \langle , \rangle, J_1, J_2, J_3)$ a quaternion–Hermitian vector space.

As Ω is $\hat{G}$-invariant, we have $\widetilde{\nabla}\Omega = 0$, so S is also a homogeneous quaternionic Kähler structure. On the other hand, in [4] we get formula (1.4) below, which furnishes explicitly the coefficients θ^a, $a = 1, 2, 3$, in Eq. (1.1). First, notice that the Lie subgroup of K generated by $\mathfrak{u}$ is a normal subgroup isomorphic to $Sp(1)$, and there exist an ideal $\mathfrak{k}_1$ of $\mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{u} \oplus \mathfrak{k}_1$, so that we can get a basis $\mathcal{B} = \{E_1, E_2, E_3, \ldots\}$ of $\mathfrak{k}$ with the basic elements of $\mathfrak{u}$ and some elements of $\mathfrak{k}_1$. We proved in [4] that the homogeneous Riemannian structure S on $M = G/K$ associated with the reductive decomposition $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ satisfies Eq. (1.1) with 1-forms θ^i, $i = 1, 2, 3$, given at $o \equiv H \in \hat{G}/H \equiv M$ by

$$\theta^i(X^*_o) = 2\alpha_i((\xi(X_{\mathfrak{p}}))_{\mathfrak{k}}), \tag{1.4}$$

for each $X \in \mathfrak{a} + \mathfrak{n}$, where $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ is the dual basis of $\mathcal{B}$.

2. The three 12-dimensional Alekseevsky spaces

We want to obtain all the homogeneous descriptions of the Alekseevsky spaces of dimension 12. We rename them as $A_{SO_0(4,3)} = SO_0(4,3)/S(O(4) \times O(3))$, $A_{SU(3,2)} = SU(3,2)/S(U(3) \times U(2))$, and $A_{Sp(3,1)} = Sp(3,1)/(Sp(3) \times Sp(1)) = \mathbb{H}\mathrm{H}(3)$. As the center of each of the corresponding full isometry groups is finite, Theorems 3 and 4 apply. The quaternionic Kähler structure of each one of these spaces is associated with a natural structure of quaternion–Hermitian vector space on the Lie algebra $\mathfrak{a}+\mathfrak{n}$ of the solvable factor AN of an Iwasawa decomposition of its full connected group of isometries. Moreover, to determine each homogeneous Riemannian structure S, we will use (1.3) and the identifications given by (1.2). In particular, for every $X \in \mathfrak{a}+\mathfrak{n}$, we will also denote by X the vector $X^*_o = (X_{\mathfrak{p}})^*_o \in T_o(M)$, and we will give the values $S_X Y = S_{X^*_o} Y^*_o$, for all X, Y in a suitable basis of $\mathfrak{a}+\mathfrak{n}$. Each such structure S, defined by a reductive decomposition $\hat{\mathfrak{g}} = \mathfrak{h}+\mathfrak{m}$, is a homogeneous quaternionic Kähler structure, and formula (1.4) will allow us to calculate directly the forms θ^a in (1.1).

2.1. The hyperbolic Grassmannian $A_{SO_0(4,3)}$

The Lie algebra of $SO_0(4,3)$ is

$$\mathfrak{so}(4,3) = \left\{ \begin{pmatrix} A & B \\ B^{\mathrm{T}} & C \end{pmatrix} \in \mathfrak{sl}(7,\mathbb{R}) : A \in \mathfrak{so}(4),\, C \in \mathfrak{so}(3) \right\}.$$

The involution τ of $\mathfrak{so}(4,3)$ given by $\tau(X) = -X^{\mathrm{T}}$ defines the Cartan decomposition $\mathfrak{so}(4,3) = \mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(3)$. We consider the subspace $\mathfrak{a}$ of $\mathfrak{p}$ defined by the matrices with real entries s_1 at the positions (45) and (54) (resp. s_2 at (36) and (63); s_3 at (72) and (27)). Then, $\mathfrak{a}$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{so}(4,3)$, and $Z_{\mathfrak{k}}(\mathfrak{a}) = \{0\}$. Let A_1, A_2 and A_3 be the elements of $\mathfrak{a}$ defined by $(s_1,s_2,s_3) = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$, respectively, which generate $\mathfrak{a}$. Let $f_j \in \mathfrak{a}^*$ such that $f_j(A_i) = \delta_{ji}$. Then, the sets of positive roots and simple roots (with respect to a suitable order in $\mathfrak{a}^*$) are $\Sigma^+ = \{f_1 \pm f_2, f_1 \pm f_3, f_2 \pm f_3, f_1, f_2, f_3\}$ and $\Pi = \{f_1 - f_2, f_2 - f_3, f_3\}$, respectively. The positive root vector spaces are given by

$$\mathfrak{g}_{f_1+f_2} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&r&-r&0&0\\ 0&0&-r&0&0&r&0\\ 0&0&-r&0&0&r&0\\ 0&0&0&r&-r&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{f_1-f_2} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&r&-r&0&0\\ 0&0&-r&0&0&-r&0\\ 0&0&-r&0&0&-r&0\\ 0&0&0&-r&r&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_{f_1+f_3} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&r&-r&0&0\\ 0&0&0&0&0&0&0\\ 0&-r&0&0&0&0&r\\ 0&-r&0&0&0&0&r\\ 0&0&0&0&0&0&0\\ 0&0&0&r&-r&0&0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{f_1-f_3} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&r&-r&0&0\\ 0&0&0&0&0&0&0\\ 0&-r&0&0&0&0&-r\\ 0&-r&0&0&0&0&-r\\ 0&0&0&0&0&0&0\\ 0&0&0&-r&r&0&0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_{f_2+f_3} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&r&0&0&-r&0\\ 0&-r&0&0&0&0&r\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-r&0&0&0&0&r\\ 0&0&r&0&0&-r&0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{f_2-f_3} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&r&0&0&-r&0\\ 0&-r&0&0&0&0&-r\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-r&0&0&0&0&-r\\ 0&0&-r&0&0&r&0 \end{pmatrix} \right\}, \tag{2.1}$$

$$\mathfrak{g}_{f_1} = \left\{ \begin{pmatrix} 0&0&0&u&-u&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -u&0&0&0&0&0&0\\ -u&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{f_2} = \left\{ \begin{pmatrix} 0&0&u&0&0&-u&0\\ 0&0&0&0&0&0&0\\ -u&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -u&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_{f_3} = \left\{ \begin{pmatrix} 0 & u & 0 & 0 & 0 & 0 & -u \\ -u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \tag{2.2}$$

where $r, u \in \mathbb{R}$. The root vector spaces for the respective opposite roots are the corresponding sets of opposite transpose matrices. For each $f = f_1 \pm f_2, f_1 \pm f_3, f_2 \pm f_3$, let X_f be the generator of $\mathfrak{g}_f$ in (2.1) obtained by putting $r = 1$ for each nonnull entry. For $f = f_1, f_2, f_3$, let U_f be the generator of $\mathfrak{g}_f$ in (2.2) obtained by putting $u = 1$ in each nonzero entry. Let also X_f and U_f be the corresponding elements of $\mathfrak{g}_f$ for the respective opposite roots $f \in \Sigma \setminus \Sigma^+$. The restricted-root space decomposition is $\mathfrak{so}(4,3) = \mathfrak{a} + \sum_{f\in\Sigma} \mathfrak{g}_f$. Now, we put $X_1 = X_{f_1+f_2}$, $Y_1 = X_{-f_1-f_2}$, $X_2 = X_{f_1-f_2}$, $Y_2 = X_{-f_1+f_2}$, $X_3 = X_{f_1+f_3}$, $Y_3 = X_{-f_1-f_3}$, $X_4 = X_{f_1-f_3}$, $Y_4 = X_{-f_1+f_3}$, $X_5 = X_{f_2+f_3}$, $Y_5 = X_{-f_2-f_3}$, $X_6 = X_{f_2-f_3}$, $Y_6 = X_{-f_2+f_3}$, $U_j = U_{f_j}$, $V_j = U_{-f_j}$, $(1 \leqslant j \leqslant 3)$. We have the Iwasawa decomposition $\mathfrak{so}(4,3) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n} = \sum_{f\in\Sigma^+} \mathfrak{g}_f = \langle X_i, U_j : 1 \leqslant i \leqslant 6; 1 \leqslant j \leqslant 3\rangle$.

The elements E_1, E_2, E_3 of $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(3)$ given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively, satisfy $[E_1, E_2] = 2E_3$, $[E_2, E_3] = 2E_1$, $[E_3, E_1] = 2E_2$, and generate a compact ideal $\mathfrak{u} \cong \mathfrak{sp}(1)$ of $\mathfrak{k}$. The isotropy representation $\mathfrak{u} \to \mathfrak{gl}(\mathfrak{p})$ defines a quaternionic Kähler structure on $A_{SO_0(4,3)}$. The action of each E_i on $\mathfrak{p}$ and the isomorphisms $\mathfrak{p} \cong \mathfrak{so}(4,3)/\mathfrak{k} \cong \mathfrak{a} + \mathfrak{n}$ define the complex structures J_i $(i = 1, 2, 3)$ acting on $\mathfrak{a} + \mathfrak{n}$. The action of each J_i on the elements of the basis of $\mathfrak{a} + \mathfrak{n}$ is given by

	A_1	A_2	A_3	U_1	U_2	U_2
J_1	$-\frac{1}{2}(X_1+X_2)$	$-\frac{1}{2}(X_1-X_2)$	$-U_3$	$-\frac{1}{2}(X_3+X_4)$	$-\frac{1}{2}(X_5+X_6)$	A_3
J_2	$-\frac{1}{2}(X_3+X_4)$	U_2	$-\frac{1}{2}(X_3-X_4)$	$\frac{1}{2}(X_1+X_2)$	$-A_2$	$-\frac{1}{2}(X_5-X_6)$
J_3	$-U_1$	$-\frac{1}{2}(X_5+X_6)$	$-\frac{1}{2}(X_5-X_6)$	A_1	$\frac{1}{2}(X_1-X_2)$	$\frac{1}{2}(X_3-X_4)$

	X_1	X_2	X_3	X_4	X_5	X_6
J_1	A_1+A_2	A_1-A_2	$\frac{1}{2}(X_5-X_6)+U_1$	$-\frac{1}{2}(X_5-X_6)+U_1$	$-\frac{1}{2}(X_3-X_4)+U_2$	$\frac{1}{2}(X_3-X_4)+U_2$
J_2	$-\frac{1}{2}(X_5+X_6)-U_1$	$\frac{1}{2}(X_5+X_6)-U_1$	A_1+A_3	A_1-A_3	$\frac{1}{2}(X_1-X_2)+U_3$	$\frac{1}{2}(X_1-X_2)-U_3$
J_3	$\frac{1}{2}(X_3+X_4)-U_2$	$\frac{1}{2}(X_3+X_4)+U_2$	$-\frac{1}{2}(X_1+X_2)-U_3$	$-\frac{1}{2}(X_1+X_2)+U_3$	A_2+A_3	A_2-A_3

We consider the scalar product $\langle , \rangle$ induced in $\mathfrak{a} + \mathfrak{n}$ through the isomorphism $\mathfrak{p} \cong \mathfrak{a} + \mathfrak{n}$ and $\frac{1}{10}B_{|\mathfrak{p}\times\mathfrak{p}}$. This product makes the basis orthogonal, with $\langle A_j, A_j\rangle = \langle U_j, U_j\rangle = 1$, $\langle X_j, X_j\rangle = 2$, and $(\mathfrak{a} + \mathfrak{n}, \langle , \rangle, J_1, J_2, J_3)$ is a quaternion–Hermitian vector space.

2.1.1. *Homogeneous descriptions of $A_{SO_0(4,3)}$ and homogeneous quaternionic Kähler structures*

The different descriptions of $A_{SO_0(4,3)}$ as a homogeneous Riemannian space will follow from the refined Langlands decompositions of the parabolic subalgebras $\mathfrak{p}_\Psi$ of $\mathfrak{so}(4,3)$, by using Theorem 4. The standard parabolic subalgebras of $\mathfrak{so}(4,3)$ are parametrized by the family of all the subsets of Π: Π, $\emptyset$, $\Psi_1 = \{f_1 - f_2, f_2 - f_3\}$, $\Psi_2 = \{f_1 - f_2, f_3\}$, $\Psi_3 = \{f_2 - f_3, f_3\}$, $\Psi_4 = \{f_1 - f_2\}$, $\Psi_5 = \{f_2 - f_3\}$, $\Psi_6 = \{f_3\}$.

For each one of the eight cases there will exist only one possible choice of the normal subgroup L of the semisimple Lie group L'_Ψ and of the subgroup E of $E'_\Psi A_\Psi$, so that $\hat{G} = LEN_\Psi$ be a connected cocompact subgroup of

Table 1

	A_1	A_2	A_3	U_1	U_2	U_3	X_1	X_2	X_3	X_4	X_5	X_6
S_{A_1}	0	0	0	0	0	0	0	0	0	0	0	0
S_{A_2}	0	0	0	0	0	0	0	0	0	0	0	0
S_{A_3}	0	0	0	0	0	0	0	0	0	0	0	0
S_{U_1}	$-U_1$	0	0	A_1	$\frac{1}{2}(X_1-X_2)$	$\frac{1}{2}(X_3-X_4)$	$-U_2$	U_2	$-U_3$	U_3	0	0
S_{U_2}	0	$-U_2$	0	$-\frac{1}{2}(X_1+X_2)$	A_2	$\frac{1}{2}(X_5-X_6)$	U_1	U_1	0	0	$-U_3$	U_3
S_{U_3}	0	0	$-U_3$	$-\frac{1}{2}(X_3+X_4)$	$-\frac{1}{2}(X_5+X_6)$	A_3	0	0	U_1	U_1	U_2	U_2
S_{X_1}	$-X_1$	$-X_1$	0	$-U_2$	U_1	0	$2A_1+2A_2$	0	$-X_6$	$-X_5$	X_4	X_3
S_{X_2}	$-X_2$	X_2	0	U_2	$-U_1$	0	0	$2A_1-2A_2$	X_5	X_6	$-X_3$	$-X_4$
S_{X_3}	$-X_3$	0	$-X_3$	$-U_3$	0	U_1	$-X_6$	X_5	$2A_1+2A_3$	0	$-X_2$	X_1
S_{X_4}	$-X_4$	0	X_4	U_3	0	$-U_1$	$-X_5$	X_6	0	$2A_1-2A_3$	X_1	$-X_2$
S_{X_5}	0	$-X_5$	$-X_5$	0	$-U_3$	U_2	X_4	X_3	$-X_2$	$-X_1$	$2A_2+2A_3$	0
S_{X_6}	0	$-X_6$	X_6	0	U_3	$-U_2$	X_3	X_4	$-X_1$	$-X_2$	0	$2A_2-2A_3$

$SO_0(4,3)$ which acts transitively on $A_{SO_0(4,3)}$. It must be $L = L'_{\Psi}$ and $E = A_{\Psi}$, and hence $\hat{G}$ coincides with the corresponding parabolic subgroup P_{Ψ}. On the other hand, the homogeneous Riemannian structures associated with the reductive decompositions obtained in a natural way from the parabolic subalgebras $\mathfrak{p}_{\Psi}$ are homogeneous quaternionic Kähler structures. We will use (1.3) to determine these structures.

The case $\Psi = \Pi$. We have $[\Psi] = \Sigma$, $\mathfrak{e}'_{\Pi} = \mathfrak{a}_{\Pi} = \mathfrak{n}_{\Pi} = \{0\}$, and $\mathfrak{p}_{\Pi} = \mathfrak{so}(4,3) + \{0\} + \{0\} + \{0\}$. The present case gives the description as a symmetric space $A_{SO_0(4,3)} \equiv SO_0(4,3)/(SO(4) \times SO(3))$. The associated reductive decomposition is the Cartan decomposition $\mathfrak{so}(4,3) = \mathfrak{k} + \mathfrak{p}$, with $\mathfrak{k} \cong \mathfrak{so}(4) \oplus \mathfrak{so}(3)$, and the corresponding homogeneous quaternionic Kähler structure is $S = 0$.

The case $\Psi = \emptyset$. The refined Langlands decomposition of the minimal parabolic subalgebra is $\mathfrak{p}_{\emptyset} = \{0\} + \{0\} + \mathfrak{a} + \mathfrak{n}$. This provides the description of $A_{SO_0(4,3)}$ as the solvable Lie group $\hat{G} = AN$, where N is the nilpotent factor in the Iwasawa decomposition of $SO_0(4,3)$. The associated reductive decomposition is $\mathfrak{a} + \mathfrak{n} = \{0\} + (\mathfrak{a} + \mathfrak{n})$, and the corresponding homogeneous quaternionic Kähler structure S is given by Table 1.

Eq. (1.1) are satisfied, with the following nonzero values of θ^i at o: $\theta^1(U_3) = \theta^1(X_1) = \theta^1(X_2) = -\theta^2(U_2) = \theta^2(X_3) = \theta^2(X_4) = \theta^3(U_1) = \theta^3(X_5) = \theta^3(X_6) = 1$.

The case $\Psi = \Psi_1$. Then $[\Psi_1] = \{\pm(f_1 - f_2), \pm(f_2 - f_3), \pm(f_1 - f_3)\}$, and $\mathfrak{p}_{\Psi_1} = \mathfrak{l}'_{\Psi_1} + \{0\} + \mathfrak{a}_{\Psi_1} + \mathfrak{n}_{\Psi_1}$, where $\mathfrak{a}_{\Psi_1} = \langle A_1 + A_2 + A_3\rangle$, $\mathfrak{n}_{\Psi_1} = \langle X_1, X_3, X_5, U_1, U_2, U_3\rangle$, and $\mathfrak{l}'_{\Psi_1} = \mathfrak{a}^{\perp}_{\Psi_1} + \langle X_2, Y_2, X_4, Y_4, X_6, Y_6\rangle$, $\mathfrak{a}^{\perp}_{\Psi_1} = \langle A_1 - A_2, A_2 - A_3\rangle$, that is

$$\mathfrak{l}'_{\Psi_1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & y_2 & y_1 & s_3 \\ 0 & -x_1 & 0 & x_3 & y_3 & s_2 & y_1 \\ 0 & -x_2 & -x_3 & 0 & s_1 & y_3 & y_2 \\ 0 & y_2 & y_3 & s_1 & 0 & -x_3 & -x_2 \\ 0 & y_1 & s_2 & y_3 & x_3 & 0 & -x_1 \\ 0 & s_3 & y_1 & y_2 & x_2 & x_1 & 0 \end{pmatrix} : \begin{array}{c} x_j, y_j, s_j \in \mathbb{R} \\ (1 \leqslant j \leqslant 3), \\ s_1 + s_2 + s_3 = 0 \end{array} \right\} \cong \mathfrak{sl}(3,\mathbb{R}).$$

We have $\hat{G} = P_{\Psi_1} \cong Sl(3,\mathbb{R})\mathbb{R}N_{\Psi_1}$, and the isotropy algebra is $\mathfrak{h} = \hat{\mathfrak{g}} \cap \mathfrak{k} = \mathfrak{l}'_{\Psi_1} \cap (\mathfrak{so}(4) \oplus \mathfrak{so}(3)) = \langle (X_2)_{\mathfrak{k}}, (X_4)_{\mathfrak{k}}, (X_6)_{\mathfrak{k}}\rangle \cong \mathfrak{so}(3)$. We have the reductive decomposition $\mathfrak{p}_{\Psi_1} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m} = \langle A_1, A_2, A_3, U_1, U_2, U_3, X_1, (X_2)_{\mathfrak{p}}, X_3, (X_4)_{\mathfrak{p}}, X_5, (X_6)_{\mathfrak{p}}\rangle$, which is associated with the homogeneous description

$A_{SO_0(4,3)} \equiv Sl(3,\mathbb{R})\mathbb{R}N_{\Psi_1}/SO(3)$. The corresponding structure S is given at o by Table 1, except that $S_{X_2}(\cdot) = S_{X_4}(\cdot) = S_{X_6}(\cdot) = 0$. Eq. (1.1) are satisfied with the following nonzero values of θ^i at o: $\theta^1(U_3) = \theta^1(X_1) = -\theta^2(U_2) = \theta^2(X_3) = \theta^3(U_1) = \theta^3(X_5) = 1$.

The case $\Psi = \Psi_2$. Then $[\Psi_2] = \{\pm(f_1 - f_2), \pm f_3\}$, $\mathfrak{p}_{\Psi_2} = \mathfrak{l}'_{\Psi_2} + \{0\} + \mathfrak{a}_{\Psi_2} + \mathfrak{n}_{\Psi_2}$, where $\mathfrak{a}_{\Psi_2} = \langle A_1 + A_2\rangle$, $\mathfrak{n}_{\Psi_2} = \langle X_1, X_3, X_4, X_5, X_6, U_1, U_2\rangle$, and $\mathfrak{l}'_{\Psi_2} = \mathfrak{a}^{\perp}_{\Psi_2} + \langle X_2, Y_2, U_3, V_3\rangle$, $a^{\perp}_{\Psi_2} = \langle A_1 - A_2, A_3\rangle$, that is

$$\mathfrak{l}'_{\Psi_2} = \left\{ \begin{pmatrix} 0 & u & 0 & 0 & 0 & 0 & v \\ -u & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & x & y & -s & 0 \\ 0 & 0 & -x & 0 & s & y & 0 \\ 0 & 0 & y & s & 0 & -x & 0 \\ 0 & 0 & -s & y & x & 0 & 0 \\ v & t & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : s, t, x, y, v \in \mathbb{R} \right\} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}).$$

Since $\hat{G} = P_{\Psi_2} \cong (Sl(2,\mathbb{R}) \times Sl(2,\mathbb{R}))\mathbb{R}N_{\Psi_2}$ and $\mathfrak{h} = \mathfrak{l}'_{\Psi_2} \cap \mathfrak{k} = \langle (U_3)_{\mathfrak{k}}, (X_2)_{\mathfrak{k}}\rangle \cong \mathfrak{so}(2) \oplus \mathfrak{so}(2)$, we have $A_{SO_0(4,3)} \equiv (Sl(2,\mathbb{R}) \times Sl(2,\mathbb{R}))\mathbb{R}N_{\Psi_2}/(SO(2) \times SO(2))$, whose natural associated reductive decomposition is $\mathfrak{p}_{\Psi_2} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m} = \langle A_1, A_2, A_3, U_1, U_2, (U_3)_{\mathfrak{p}}, X_1, (X_2)_{\mathfrak{p}}, X_3, X_4, X_5, X_6\rangle$. Its structure S is given at o by Table 1, except that $S_{U_3}(\cdot) = S_{X_2}(\cdot) = 0$. Eq. (1.1) are satisfied with the following nonzero values of θ^i at o: $\theta^1(X_1) = -\theta^2(U_2) = \theta^2(X_3) = \theta^2(X_4) = \theta^3(U_1) = \theta^3(X_5) = \theta^3(X_6) = 1$.

The case $\Psi = \Psi_3$. Then $[\Psi_3] = \{\pm f_2 \pm f_3, \pm f_2, \pm f_3\}$ and $\mathfrak{p}_{\Psi_3} = \mathfrak{l}'_{\Psi_3} + \{0\} + \mathfrak{a}_{\Psi_3} + \mathfrak{n}_{\Psi_3}$, where $\mathfrak{a}_{\Psi_3} = \langle A_1\rangle$, $\mathfrak{n}_{\Psi_3} = \langle X_1, X_2, X_3, X_4, U_1\rangle$, and $\mathfrak{l}'_{\Psi_3} = \mathfrak{a}^{\perp}_{\Psi_3} + \langle X_5, Y_5, X_6, Y_6, U_2, V_2, U_3, V_3\rangle$, $\mathfrak{a}^{\perp}_{\Psi_3} = \langle A_2, A_3\rangle$, that is

$$\mathfrak{l}'_{\Psi_3} = \left\{ \begin{pmatrix} 0 & u_1 & u_2 & 0 & 0 & v_2 & v_1 \\ -u_1 & 0 & x_1 & 0 & 0 & y_1 & t \\ -u_2 & -x_1 & 0 & 0 & 0 & s & y_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & y_1 & s & 0 & 0 & 0 & x_2 \\ v_1 & t & y_2 & 0 & 0 & -x_2 & 0 \end{pmatrix} : \begin{matrix} x_j, y_j, u_j, v_j, \\ s, t \in \mathbb{R} \\ (1 \leqslant j \leqslant 3) \end{matrix} \right\} \cong \mathfrak{so}(3,2).$$

We have $\hat{G} = P_{\Psi_3} \cong SO_0(3,2)\mathbb{R}N_{\Psi_3}$, and $\mathfrak{h} = \mathfrak{l}'_{\Psi_3} \cap \mathfrak{k} = \langle (U_2)_{\mathfrak{k}}, (U_3)_{\mathfrak{k}}, (X_5)_{\mathfrak{k}}, (X_6)_{\mathfrak{k}}\rangle \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2)$, then $A_{SO_0(4,3)} \equiv SO_0(3,2)\mathbb{R}N_{\Psi_3}/(SO(3) \times SO(2))$, and $\mathfrak{p}_{\Psi_3} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m} = \langle A_1, A_2, A_3, U_1, (U_2)_{\mathfrak{p}}, (U_3)_{\mathfrak{p}}, X_1, X_2, X_3, X_4, (X_5)_{\mathfrak{p}}, (X_6)_{\mathfrak{p}}\rangle$. The corresponding structure S is given at o by Table 1, except that $S_{U_2}(\cdot) = S_{U_3}(\cdot) = S_{X_5}(\cdot) = S_{X_6}(\cdot) = 0$. Eq. (1.1) are satisfied with the following nonzero values of θ^i at o: $\theta^1(X_1) = \theta^1(X_2) = \theta^2(X_3) = \theta^2(X_4) = \theta^3(U_1) = 1$.

The case $\Psi = \Psi_4$. Then $[\Psi_4] = \{\pm(f_1 - f_2)\}$ and $\mathfrak{p}_{\Psi_4} = \mathfrak{l}'_{\Psi_4} + \{0\} + \mathfrak{a}_{\Psi_4} + \mathfrak{n}_{\Psi_4}$, where $\mathfrak{a}_{\Psi_4} = \langle A_1 + A_2, A_3\rangle$, $\mathfrak{n}_{\Psi_4} = \langle X_1, X_3, X_4, X_5, X_6, U_1, U_2, U_3\rangle$, and $\mathfrak{l}'_{\Psi_4} = \mathfrak{a}^{\perp}_{\Psi_4} + \langle X_2, Y_2\rangle$, $\mathfrak{a}^{\perp}_{\Psi_4} = \langle A_1 - A_2\rangle$, that is $\mathfrak{l}'_{\Psi_4} \cong \mathfrak{sl}(2,\mathbb{R})$. This gives $A_{SO_0(4,3)} \equiv Sl(2,\mathbb{R})\mathbb{R}^2N_{\Psi_4}/SO(2)$, with the reductive decomposition $\mathfrak{p}_{\Psi_4} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h} = \langle (X_2)_{\mathfrak{k}}\rangle \cong \mathfrak{so}(2)$, and $\mathfrak{m} = \langle A_1, A_2, A_3, U_1, U_2, U_3, X_1, (X_2)_{\mathfrak{p}}, X_3, X_4, X_5, X_6\rangle$. The corresponding structure S is given at o by Table 1, except that $S_{X_2}(\cdot) = 0$. Eq. (1.1) are satisfied with θ^i at o as in $\Psi = \emptyset$, except that $\theta^1(X_2) = 0$.

The case $\Psi = \Psi_5$. Then $[\Psi_5] = \{\pm(f_2 - f_3)\}$ and $\mathfrak{p}_{\Psi_5} = \mathfrak{l}'_{\Psi_5} + \{0\} + \mathfrak{a}_{\Psi_5} + \mathfrak{n}_{\Psi_5}$, where $\mathfrak{a}_{\Psi_5} = \langle A_1, A_2 + A_3\rangle$, $\mathfrak{n}_{\Psi_5} = \langle X_1, X_2, X_3, X_4, X_5, U_1, U_2, U_3\rangle$, and $\mathfrak{l}'_{\Psi_5} = \mathfrak{a}^{\perp}_{\Psi_5} + \langle X_6, Y_6\rangle$, $\mathfrak{a}^{\perp}_{\Psi_5} = \langle A_2 - A_3\rangle$, that is $\mathfrak{l}'_{\Psi_5} \cong \mathfrak{sl}(2,\mathbb{R})$. This gives $A_{SO_0(4,3)} \equiv Sl(2,\mathbb{R})\mathbb{R}^2N_{\Psi_5}/SO(2)$, with $\mathfrak{p}_{\Psi_5} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h} = \langle (X_6)_{\mathfrak{k}}\rangle \cong \mathfrak{so}(2)$, and $\mathfrak{m} = \langle A_1, A_2, A_3, U_1, U_2, U_3, X_1, X_2, X_3, X_4, X_5, (X_6)_{\mathfrak{p}}\rangle$. The corresponding structure S is given at o by Table 1, except that $S_{X_6}(\cdot) = 0$. Eq. (1.1) are satisfied with θ^i at o as in $\Psi = \emptyset$, except that $\theta^3(X_6) = 0$.

The case $\Psi = \Psi_6$. Then $[\Psi_6] = \{\pm f_3\}$ and $\mathfrak{p}_{\Psi_6} = \mathfrak{l}'_{\Psi_6} + \{0\} + \mathfrak{a}_{\Psi_6} + \mathfrak{n}_{\Psi_6}$, where $\mathfrak{a}_{\Psi_6} = \langle A_1, A_2\rangle$, $\mathfrak{n}_{\Psi_6} = \langle X_1, X_2, X_3, X_4, X_5, X_6, U_1, U_2\rangle$, and $\mathfrak{l}'_{\Psi_6} = \mathfrak{a}^{\perp}_{\Psi_6} + \langle U_3, V_3\rangle$, $\mathfrak{a}^{\perp}_{\Psi_6} = \langle A_3\rangle$, that is $\mathfrak{l}'_{\Psi_6} \cong \mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$. This gives $A_{SO_0(4,3)} \equiv Sl(2,\mathbb{R})\mathbb{R}^2N_{\Psi_6}/SO(2)$, and $\mathfrak{p}_{\Psi_6} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h} = \langle (U_3)_{\mathfrak{k}}\rangle \cong \mathfrak{so}(2), \mathfrak{m} = \langle A_1, A_2, A_3, U_1, U_2, (U_3)_{\mathfrak{p}}, X_1, X_2, X_3, X_4, X_5, X_6\rangle$. The corresponding structure S is given at o by Table 1, except that $S_{U_3}(\cdot) = 0$. Eq. (1.1) are satisfied with θ^i at o as in $\Psi = \emptyset$, except that $\theta^1(U_3) = 0$.

Table 2

	ϑ	$F(T-T^{\vartheta})_{XYZ}$	$(T-T^{\vartheta})_{XYZ}$	XYZ
$\emptyset$	$\frac{1}{28}\langle 7A_1+4A_2+A_3,\cdot\rangle$	$-\frac{16}{7}$	$\frac{5}{14}$	$U_1X_2U_2$
Ψ_1	$\frac{1}{7}\langle A_1+A_2+A_3,\cdot\rangle$	$-\frac{25}{7}$	$\frac{5}{7}$	$X_1U_2U_1$
Ψ_2	$\frac{11}{56}\langle A_1+A_2,\cdot\rangle$	$\frac{5}{14}$	$\frac{17}{56}$	$U_2X_2X_1$
Ψ_3	$\frac{1}{4}\langle A_1,\cdot\rangle$	5	$-\frac{5}{4}$	$X_3X_1X_6$
Ψ_4	$\frac{1}{56}\langle 11A_1+11A_2+2A_3,\cdot\rangle$	$\frac{173}{56}$	$-\frac{71}{56}$	$X_3X_1X_6$
Ψ_5	$\frac{1}{56}\langle 14A_1+5A_2+5A_3,\cdot\rangle$	$-\frac{3}{7}$	$\frac{23}{56}$	$U_1X_2U_2$
Ψ_6	$\frac{1}{28}\langle 7A_1+4A_2,\cdot\rangle$	$-\frac{12}{7}$	$\frac{25}{28}$	$X_2U_1U_2$

Now, we know that such a structure S decomposes as $S=\Theta+T$, with $\Theta\in\mathcal{QK}_{12}$, i.e., such that $\Theta_XY=\frac{1}{2}\sum_{a=1}^{3}\theta^a(X)J_aY$, and $T\in\mathcal{QK}_{345}$. The condition for the tensor Θ to be in $\mathcal{QK}_1$ is $\theta^a=\theta\circ J_a$, $a=1,2,3$, for some 1-form θ. Then, as some calculations show, we have in all the cases that $\Theta\in\mathcal{QK}_{12}\setminus\mathcal{QK}_1\cup\mathcal{QK}_2$, except for Ψ_3, where $\Theta\in\mathcal{QK}_1$ with corresponding 1-form $\theta=\langle A_1,\cdot\rangle$. Further, since $\dim M=12$, the 1-form defining the $\mathcal{QK}_3$-component of T (see Theorem 2) is given by $\vartheta=\frac{1}{14}c_{12}$. The respective values of ϑ are given by Table 2, so the $\mathcal{QK}_3$-component of T never vanishes in these cases.

Let $F\colon\hat{\mathcal{V}}\to\hat{\mathcal{V}}$ be the operator defined by $F(T)_{XYZ}=T_{ZXY}+T_{YZX}+\sum_{a=1}^{3}(T_{J_aZXJ_aY}+T_{J_aYJ_aZX})$, having eigenvalues 2 and -4, and respective eigenspaces $\mathcal{QK}_{34}$ and $\mathcal{QK}_5$ (see Theorem 2). Consider $T^{\vartheta}\in\mathcal{QK}_3$ given by $T^{\vartheta}_{XYZ}=\langle X,Y\rangle\vartheta(Z)-\langle X,Z\rangle\vartheta(Y)+\sum_{a=1}^{3}(\langle X,J_aY\rangle\vartheta(J_aZ)-\langle X,J_aZ\rangle\vartheta(J_aY))$, where ϑ is the above-mentioned 1-form. Then $T-T^{\vartheta}\in\mathcal{QK}_{45}$ and we get $F(T-T^{\vartheta})_{XYZ}=F(T)_{XYZ}-2T^{\vartheta}_{XYZ}$. Computing, we have the values for $F(T-T^{\vartheta})_{XYZ}$ and $(T-T^{\vartheta})_{XYZ}$ given in Table 2, where also the vectors X,Y,Z chosen in each case appear. Hence, the tensor S has a nonzero component in each primitive subspace $\mathcal{QK}_i$, for $i=1,\ldots,5$, except for Ψ_3. In this case, as the result for the choice of vectors X_2,U_1,U_2 suggests, and a computation with Maple shows, we obtain $T-T^{\vartheta}\in\mathcal{QK}_5$, so $S\in\mathcal{QK}_{135}$.

2.2. *The complex hyperbolic Grassmannian $A_{SU(3,2)}$*

The Lie algebra of $SU(3,2)$ is

$$\mathfrak{su}(3,2)=\left\{\begin{pmatrix}A & B\\ \bar{B}^{\mathrm{T}} & C\end{pmatrix}\in\mathfrak{sl}(5,\mathbb{C}):A\in\mathfrak{u}(3),\,C\in\mathfrak{u}(2)\right\}.$$

The involution τ of $\mathfrak{su}(3,2)$ given by $\tau(X)=-\bar{X}^{\mathrm{T}}$ defines the Cartan decomposition $\mathfrak{su}(3,2)=\mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k}=\mathfrak{s}(\mathfrak{u}(3)\oplus\mathfrak{u}(2))$. We consider the subspace $\mathfrak{a}$ of $\mathfrak{p}$ defined by the matrices with real entries s_1 at the positions (34) and (43) and s_2 at (25) and (52). Then $\mathfrak{a}$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{su}(3,2)$. Let A_1 and A_2 be the generators of $\mathfrak{a}$ defined by $(s_1,s_2)=(1,0)$ and $(0,1)$, respectively. Let f_1 and f_2 be the elements of $\mathfrak{a}^*$ given by $f_j(A_i)=\delta_{ji}$. Then, the set of positive roots and simple roots (with respect to a suitable order in $\mathfrak{a}^*$) are $\Sigma^+=\{f_1\pm f_2,2f_1,2f_2,f_1,f_2\}$ and $\Pi=\{f_1-f_2,f_2\}$, respectively. The positive root vector spaces are

$$\mathfrak{g}_{f_1+f_2}=\left\{\begin{pmatrix}0&0&0&0&0\\0&0&z&-z&0\\0&-\bar{z}&0&0&\bar{z}\\0&-\bar{z}&0&0&\bar{z}\\0&0&z&-z&0\end{pmatrix}\right\},\qquad \mathfrak{g}_{f_1-f_2}=\left\{\begin{pmatrix}0&0&0&0&0\\0&0&z&-z&0\\0&-\bar{z}&0&0&-\bar{z}\\0&-\bar{z}&0&0&-\bar{z}\\0&0&-z&z&0\end{pmatrix}\right\},\tag{2.3}$$

$$\mathfrak{g}_{2f_1}=\left\{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&ix&-ix&0\\0&0&ix&-ix&0\\0&0&0&0&0\end{pmatrix}\right\},\qquad \mathfrak{g}_{2f_2}=\left\{\begin{pmatrix}0&0&0&0&0\\0&ix&0&0&-ix\\0&0&0&0&0\\0&0&0&0&0\\0&ix&0&0&-ix\end{pmatrix}\right\},\tag{2.4}$$

$$\mathfrak{g}_{f_1} = \left\{ \begin{pmatrix} 0 & 0 & w & -w & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\bar{w} & 0 & 0 & 0 & 0 \\ -\bar{w} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{g}_{f_2} = \left\{ \begin{pmatrix} 0 & w & 0 & 0 & -w \\ -\bar{w} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\bar{w} & 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \tag{2.5}$$

where $x \in \mathbb{R}$, $z, w \in \mathbb{C}$. The root vector spaces for the respective opposite roots are the corresponding sets of opposite conjugate transpose matrices. For each $f = f_1 \pm f_2$, let X_f and X'_f be the generators of $\mathfrak{g}_f$ in (2.3) obtained by putting $z = 1$ and $z = i$, respectively, for each nonnull entry. For $f = 2f_j$ $(1 \leq j \leq 2)$, let U_f be the generator of $\mathfrak{g}_f$ in (2.4) obtained by taking $x = 1$ in each nonzero entry. For $f = f_j$ $(1 \leq j \leq 2)$, let P_f and P'_f be the generators of $\mathfrak{g}_f$ in (2.5) obtained by putting $w = 1$ and $w = i$, respectively, for each nonnull entry. Let also X_f, X'_f, U_f, P_f, P'_f be the corresponding elements of $\mathfrak{g}_f$ for the respective opposite roots $f \in \Sigma \setminus \Sigma^+$. We put (for $j = 1, 2$) $X_1 = X_{f_1+f_2}$, $Y_1 = X_{-f_1-f_2}$, $X'_1 = X'_{f_1+f_2}$, $Y'_1 = X_{-f_1-f_2}$, $X_2 = X_{f_1-f_2}$, $Y_2 = X_{-f_1+f_2}$, $X'_2 = X'_{f_1-f_2}$, $Y'_2 = X_{-f_1+f_2}$, $U_j = U_{2f_j}$, $V_j = U_{-2f_j}$, $P_j = P_{f_j}$, $Q_j = P_{-f_j}$, $P'_j = P'_{f_j}$, $Q'_j = P'_{-f_j}$.

The centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is $Z_{\mathfrak{k}}(\mathfrak{a}) = \{i \cdot \mathrm{diag}(r, s, t, t, s) : r, s, t \in \mathbb{R}, r + 2s + 2t = 0\}$. Then $C_1 = \mathrm{diag}(2i, 0, -i, -i, 0)$ and $C_2 = \mathrm{diag}(2i, -i, 0, 0, -i)$ generate $Z_{\mathfrak{k}}(\mathfrak{a})$, and $Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} = \langle C_1, C_2, A_1, A_2 \rangle$ is a Cartan subalgebra of $\mathfrak{su}(3, 2)$. We so have the restricted-root space decomposition $\mathfrak{su}(3, 2) = (Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}) + \sum_{f \in \Sigma} \mathfrak{g}_f$. We also have the Iwasawa decomposition $\mathfrak{su}(3, 2) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n} = \sum_{f \in \Sigma^+} \mathfrak{g}_f = \langle X_j, X'_j, U_j, P_j, P'_j : j = 1, 2 \rangle$.

The elements

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & i & 0 \end{pmatrix},$$

of $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(2)) \subset \mathfrak{su}(3, 2)$ satisfy $[E_1, E_2] = 2E_3$, $[E_2, E_3] = 2E_1$, $[E_3, E_1] = 2E_2$, and the compact subalgebra $\mathfrak{u} \cong \mathfrak{sp}(1)$ generated by $\{E_1, E_2, E_3\}$ is an ideal of $\mathfrak{k}$. The isotropy representation $\mathfrak{u} \to \mathfrak{gl}(\mathfrak{p})$ defines a quaternionic Kähler structure on $A_{SU(3,2)}$. The action of each E_i on $\mathfrak{p}$ and the isomorphisms $\mathfrak{p} \cong \mathfrak{su}(3, 2)/\mathfrak{k} \cong \mathfrak{a} + \mathfrak{n}$ determine the complex structures J_i $(i = 1, 2, 3)$ acting on $\mathfrak{a} + \mathfrak{n}$. The action on the elements A_j, X_j, X'_j, U_j, P_j, P'_j, $(j = 1, 2)$ of the basis of $\mathfrak{a} + \mathfrak{n}$ is given by

	A_1	A_2	X_1	X'_1	X_2	X'_2	U_1	U_2	P_1	P'_1	P_2	P'_2
J_1	$-U_1$	U_2	X'_1	$-X_1$	X'_2	$-X_2$	A_1	$-A_2$	P'_1	$-P_1$	$-P'_2$	P_2
J_2	$\frac{1}{2}(X_1 - X_2)$	$\frac{1}{2}(X_1 + X_2)$	$-A_1 -A_2$	$-U_1 +U_2$	$A_1 - A_2$	$U_1 +U_2$	$\frac{1}{2}(X'_1 - X'_2)$	$-\frac{1}{2}(X'_1 + X'_2)$	P_2	P'_2	$-P_1$	$-P'_1$
J_3	$\frac{1}{2}(X'_1 - X'_2)$	$\frac{1}{2}(X'_1 + X'_2)$	$U_1 - U_2$	$-A_1 -A_2$	$-U_1 -U_2$	$A_1 -A_2$	$-\frac{1}{2}(X_1 - X_2)$	$\frac{1}{2}(X_1 + X_2)$	$-P'_2$	P_2	$-P'_1$	P_1

We consider the scalar product $\langle , \rangle$ induced in $\mathfrak{a} + \mathfrak{n}$ by the isomorphism $\mathfrak{p} \cong \mathfrak{a} + \mathfrak{n}$ and $\frac{1}{20} B_{|\mathfrak{p} \times \mathfrak{p}}$. This product makes the basis orthogonal, with $\langle A_j, A_j \rangle = \langle U_j, U_j \rangle = \langle P_j, P_j \rangle = \langle P'_j, P'_j \rangle = 1$, $\langle X_j, X_j \rangle = \langle X'_j, X'_j \rangle = 2$, and $(\mathfrak{a} + \mathfrak{n}, \langle , \rangle, J_1, J_2, J_3)$ is a quaternion–Hermitian vector space.

2.2.1. Homogeneous descriptions of $A_{SU(3,2)}$ and homogeneous quaternionic Kähler structures

By using Theorems 3 and 4, and from the refined Langlands decompositions of the parabolic subalgebras of $\mathfrak{su}(3, 2)$, we will now obtain the homogeneous descriptions of $A_{SU(3,2)}$. The standard parabolic subalgebras of $\mathfrak{su}(3, 2)$ are parametrized by the subsets Π, $\emptyset$, $\Psi_1 = \{f_1 - f_2\}$ and $\Psi_2 = \{f_2\}$ of Π.

The case $\Psi = \Pi$. We have $[\Psi] = \Sigma$, and $\mathfrak{e}'_\Pi = \mathfrak{a}_\Pi = \mathfrak{n}_\Pi = \{0\}$, so the refined Langlands decomposition is $\mathfrak{p}_\Pi = \mathfrak{su}(3, 2) + \{0\} + \{0\} + \{0\}$. The only transitive action coming from $\Psi = \Pi$ is that of the full isometry group $SU(3, 2)$, and we have the description of $A_{SU(3,2)}$ as the symmetric space $SU(3, 2)/S(U(3) \times U(2))$. The

Table 3

	A_1	A_2	X_1	X_1'	X_2	X_2'	U_1	U_2	P_1	P_1'	P_2	P_2'
S_{A_1}	0	0	$\lambda_1 X_1' - \lambda_2 X_1'$	$\lambda_2 X_1 - \lambda_1 X_1$	$\lambda_1 X_2' - \lambda_2 X_2'$	$\lambda_2 X_2 - \lambda_1 X_2$	0	0	$3\lambda_1 P_1' + 2\lambda_2 P_1'$	$-3\lambda_1 P_1 - 2\lambda_2 P_1$	$2\lambda_1 P_2' + 3\lambda_2 P_2'$	$-2\lambda_1 P_2 - 3\lambda_2 P_2$
S_{A_2}	0	0	$\mu_1 X_1' - \mu_2 X_1'$	$\mu_2 X_1 - \mu_1 X_1$	$\mu_1 X_2' - \mu_2 X_2'$	$\mu_2 X_2 - \mu_1 X_2$	0	0	$3\mu_1 P_1' + 2\mu_2 P_1'$	$-3\mu_1 P_1 - 2\mu_2 P_1$	$2\mu_1 P_2' + 3\mu_2 P_2'$	$-2\mu_1 P_2 - 3\mu_2 P_2$
S_{U_1}	$-2U_1$	0	X_2'	$-X_2$	X_1'	$-X_1$	$2A_1$	0	P_1'	$-P_1$	0	0
S_{U_2}	0	$-2U_2$	X_2'	$-X_2$	X_1'	$-X_1$	0	$2A_2$	0	0	P_2'	$-P_2$
S_{X_1}	$-X_1$	$-X_1$	$2A_1 + 2A_2$	0	0	$-2U_1 - 2U_2$	X_2'	X_2'	$-P_2$	$-P_2'$	P_1	P_1'
$S_{X_1'}$	$-X_1'$	$-X_1'$	0	$2A_1 + 2A_2$	$2U_1 + 2U_2$	0	$-X_2$	$-X_2$	P_2'	$-P_2$	P_1'	$-P_1$
S_{X_2}	$-X_2$	X_2	0	$-2U_1 + 2U_2$	$2A_1 - 2A_2$	0	X_1'	$-X_1'$	P_2	P_2'	$-P_1$	$-P_1'$
$S_{X_2'}$	$-X_2'$	X_2'	$2U_1 - 2U_2$	0	0	$2A_1 - 2A_2$	$-X_1$	X_1	$-P_2'$	P_2	$-P_1'$	P_1
S_{P_1}	$-P_1$	0	P_2	P_2'	P_2	$-P_2'$	P_1'	0	A_1	$-U_1$	$\frac{1}{2}(X_1 - X_2)$	$\frac{1}{2}(X_2' - X_1')$
$S_{P_1'}$	$-P_1'$	0	P_2'	$-P_2$	P_2'	P_2	$-P_1$	0	U_1	A_1	$\frac{1}{2}(X_1' - X_2')$	$\frac{1}{2}(X_1 - X_2)$
S_{P_2}	0	$-P_2$	P_1	P_1'	P_1	P_1'	0	P_2'	$-\frac{1}{2}(X_1 + X_2)$	$-\frac{1}{2}(X_1' + X_2')$	A_2	$-U_2$
$S_{P_2'}$	0	$-P_2'$	P_1'	$-P_1$	P_1'	$-P_1$	0	$-P_2$	$\frac{1}{2}(X_1' + X_2')$	$-\frac{1}{2}(X_1 + X_2)$	U_2	A_2

associated reductive decomposition is the Cartan decomposition $\mathfrak{su}(3,2) = \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(2)) + \mathfrak{p}$, and the corresponding homogeneous quaternionic Kähler structure is $S = 0$.

The case $\Psi = \emptyset$. Then $\mathfrak{l}'_{\emptyset} = \{0\}$, $\mathfrak{e}'_{\emptyset} = Z_{\mathfrak{k}}(\mathfrak{a})$, $\mathfrak{a}_{\emptyset} = \mathfrak{a}$, so the refined Langlands decomposition of the minimal parabolic subalgebra is $\mathfrak{p}_{\emptyset} = \{0\} + Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} + \mathfrak{n}$. For each connected closed subgroup E of $E'A \cong U(1) \times U(1) \times \mathbb{R}^2$ we get a cocompact subgroup $\hat{G} = EN$ of $SU(3,2)$, where N is the nilpotent factor in the Iwasawa decomposition of $SU(3,2)$. In order to get a transitive action it is sufficient that the projection of $\mathfrak{e} \subset Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} = \langle C_1, C_2, A_1, A_2 \rangle$ to $\mathfrak{a}$ be surjective. There are infinitely many possible choices of such a subspace $\mathfrak{e}$.

If $\dim \mathfrak{e} = 2$, then $\mathfrak{e} = \mathfrak{e}_{\lambda,\mu} = \langle \lambda_1 C_1 + \lambda_2 C_2 + A_1, \mu_1 C_1 + \mu_2 C_2 + A_2 \rangle$, for some $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, and it generates the Lie subgroup $E_{\lambda,\mu} \cong \mathbb{R}^2$ of $SU(3,2)$ such that $\hat{G} = E_{\lambda,\mu} N$ acts simply transitively on $A_{SU(3,2)}$. In particular, the choice $\mathfrak{e} = \mathfrak{a}$ gives the usual description of $A_{SU(3,2)}$ as the solvable Lie group AN. The reductive decomposition associated with the description $A_{SU(3,2)} = E_{\lambda,\mu} N$ is $\hat{\mathfrak{g}}^{\lambda,\mu} = \{0\} + \hat{\mathfrak{g}}^{\lambda,\mu}$, where $\hat{\mathfrak{g}}^{\lambda,\mu} = \langle \lambda_1 C_1 + \lambda_2 C_2 + A_1, \mu_1 C_1 + \mu_2 C_2 + A_2, X_j, X_j', U_j, P_j, P_j' : j = 1, 2 \rangle$. Then we have a four-parameter family of homogeneous Riemannian structures $S^{\lambda,\mu}$ corresponding to the family of reductive decompositions $\hat{\mathfrak{g}}^{\lambda,\mu} = \{0\} + \hat{\mathfrak{g}}^{\lambda,\mu}$. With the identifications in (1.2), $S = S^{\lambda,\mu}$ is given at o by Table 3.

Eq. (1.1) are satisfied with the following nonzero values of θ^i at o: $\theta^1(A_1) = \lambda_1 - \lambda_2$, $\theta^1(A_2) = \mu_1 - \mu_2$, $\theta^1(U_1) = -\theta^1(U_2) = 1$, $\theta^2(X_1) = -\theta^2(X_2) = \theta^3(X_1') = -\theta^3(X_2') = -2$. We have $S = \Theta + T$. It is easily seen that $\Theta \in \mathcal{QK}_{12} \setminus \mathcal{QK}_1 \cup \mathcal{QK}_2$. Since moreover we have $\vartheta = \frac{1}{28}\langle 11A_1 + 7A_2 + (\lambda_1 - \lambda_2)U_1 - (\mu_1 - \mu_2)U_2, \cdot \rangle$, $(T - T^{\vartheta})_{X_2 U_2 X_1'} = -\frac{5}{14}$, and $F(T - T^{\vartheta})_{X_2 U_2 X_1'} = -\frac{12}{7}$, the tensor S has a nonzero component in each primitive subspace $\mathcal{QK}_i$, $i, \dots, 5$.

If $\dim \mathfrak{e} = 3$, we can write $\mathfrak{e} = \langle \nu_1 C_1 + \nu_2 C_2, \lambda_1 C_1 + \lambda_2 C_2 + A_1, \mu_1 C_1 + \mu_2 C_2 + A_2 \rangle$, where $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$. The corresponding reductive decomposition is $\hat{\mathfrak{g}}_{\nu}^{\lambda,\mu} = \mathfrak{h}_{\nu} + \mathfrak{m}^{\lambda,\mu}$, where $\mathfrak{h}_{\nu} = \langle \nu_1 C_1 + \nu_2 C_2 \rangle \cong \mathfrak{u}(1)$ and $\mathfrak{m}^{\lambda,\mu} = \langle \lambda_1 C_1 + \lambda_2 C_2 + A_1, \mu_1 C_1 + \mu_2 C_2 + A_2, X_j, X_j', U_j, P_j, P_j' : j = 1, 2 \rangle$, which gives $A_{SU(3,2)} \equiv (U(1) \times \mathbb{R}^2)N/U(1)$. The associated homogeneous quaternionic Kähler structures are the structures $S^{\lambda,\mu}$ above.

If $\dim \mathfrak{e} = 4$, the reductive decomposition is $\hat{\mathfrak{g}}' = \mathfrak{h}' + \mathfrak{m}'$, where $\hat{\mathfrak{g}}' = \mathfrak{p}_{\emptyset}$, $\mathfrak{h}' = Z_{\mathfrak{k}}(\mathfrak{a})$ and $\mathfrak{m}' = \mathfrak{a} + \mathfrak{n}$, which gives the description $A_{SU(3,2)} \equiv (U(1) \times U(1) \times \mathbb{R}^2)N/(U(1) \times U(1))$. The associated structure S' coincides with the above structure $S^{\lambda,\mu}$, for $\lambda = \mu = 0$.

The case $\Psi = \Psi_1$. Then $[\Psi_1] = \{\pm(f_1 - f_2)\}$ and $\mathfrak{p}_{\Psi_1} = \mathfrak{l}'_{\Psi_1} + \mathfrak{e}'_{\Psi_1} + \mathfrak{a}_{\Psi_1} + \mathfrak{n}_{\Psi_1}$, with $\mathfrak{e}'_{\Psi_1} = i\,\mathbb{R}\cdot\mathrm{diag}(-4, 1, 1, 1, 1) = \langle C_1 + C_2\rangle$, $\mathfrak{a}_{\Psi_1} = \langle A_1 + A_2\rangle$, $\mathfrak{n}_{\Psi_1} = \langle X_1, X'_1, U_1, U_2, P_1, P'_1, P_2, P'_2\rangle$, and $\mathfrak{l}'_{\Psi_1} = \langle A_1 - A_2\rangle + \mathfrak{e}'^{\perp}_{\Psi_1} + \langle X_2, Y_2, X'_2, Y'_2\rangle$, $\mathfrak{e}'^{\perp}_{\Psi_1} = \langle C_1 - C_2\rangle \subset Z_{\mathfrak{k}}(\mathfrak{a})$, and we have

$$\mathfrak{l}'_{\Psi_1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & ir & z & w & s \\ 0 & -\bar{z} & -ir & -s & \bar{w} \\ 0 & \bar{w} & -s & -ir & -\bar{z} \\ 0 & s & w & z & ir \end{pmatrix} : r, s \in \mathbb{R}, z, w \in \mathbb{C} \right\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

For each connected closed subgroup E of P_{Ψ_1} whose Lie algebra $\mathfrak{e}$ is a nontrivial subspace of $\mathfrak{e}'_{\Psi_1} + \mathfrak{a}_{\Psi_1}$, we get a cocompact subgroup $\hat{G} = LEN_{\Psi_1}$ of $SU(3, 2)$, with $L \cong Sl(2, \mathbb{C})$. If $\mathfrak{e} \neq \mathfrak{e}'_{\Psi_1}$, then $\hat{G}$ acts transitively on $A_{SU(3,2)}$.

If $\dim \mathfrak{e} = 1$, we get $A_{SU(3,2)} \equiv Sl(2, \mathbb{C})\mathbb{R}N_{\Psi_1}/SU(2)$. For each $\lambda \in \mathbb{R}$, we have a subspace $\mathfrak{e} = \mathfrak{e}_\lambda$ of $\mathfrak{e}'_{\Psi_1} + \mathfrak{a}_{\Psi_1} = \langle C_1 + C_2, A_1 + A_2\rangle$, generated by $\lambda(C_1 + C_2) + A_1 + A_2$. One has the one-parameter family of reductive decompositions $\hat{\mathfrak{g}}^\lambda = \mathfrak{h} + \mathfrak{m}^\lambda$, where the isotropy algebra is $\mathfrak{h} = \hat{\mathfrak{g}} \cap \mathfrak{k} = \mathfrak{l}'_{\Psi_1} \cap \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(2)) = \langle C_1 - C_2, (X_2)_{\mathfrak{k}}, (X'_2)_{\mathfrak{k}}\rangle \cong \mathfrak{su}(2)$ and $\mathfrak{m}^\lambda = \langle \lambda(C_1 + C_2) + A_1, \lambda(C_1 + C_2) + A_2, X_1, X'_1, (X_2)_{\mathfrak{p}}, (X'_2)_{\mathfrak{p}}, U_1, U_2, P_1, P'_1, P_2, P'_2\rangle$. The associated one-parameter family of structures $S = S^\lambda$ is given at o by the values in Table 3 except that $S_{X_1}(\cdot) = S_{X_2}(\cdot) = 0$, and with $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \lambda$. Eq. (1.1) are satisfied with the following nonzero values of θ^i at o: $\theta^1(U_1) = -\theta^1(U_2) = 1$, $\theta^2(X_1) = \theta^3(X'_1) = -2$.

We have $S = \Theta + T$. It is easily seen that $\Theta \in \mathcal{QK}_1$ with associated form $\theta = \langle A_1 + A_2, \cdot\rangle$. As for the specific type of T inside $\mathcal{QK}_{345}$, we first have that, as $\vartheta = \frac{9}{28}\langle A_1 + A_2, \cdot\rangle$, the $\mathcal{QK}_3$-component of T does not vanish. Computing as before, we get $(T - T^\vartheta)_{U_1 X'_2 X_1} = -\frac{19}{14}$ and $F(T - T^\vartheta)_{U_1 X'_2 X_1} = \frac{44}{7}$. Hence $T - T^\vartheta \in \mathcal{QK}_{45} \setminus \mathcal{QK}_4 \cup \mathcal{QK}_5$. Hence $S \in \mathcal{QK}_{1345}$.

If $\dim \mathfrak{e} = 2$, that is, $\mathfrak{e} = \langle C_1 + C_2, A_1 + A_2\rangle \cong \mathfrak{u}(1) \oplus \mathbb{R}$, we have the reductive decomposition $\hat{\mathfrak{g}}' = \mathfrak{h}' + \mathfrak{m}'$, where $\mathfrak{h}' = (\mathfrak{l}'_{\Psi_1} + \mathfrak{e}'_{\Psi_1}) \cap \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(2)) = \langle C_1, C_2, (X_2)_{\mathfrak{k}}, (X'_2)_{\mathfrak{k}}\rangle \cong \mathfrak{u}(1) \oplus \mathfrak{su}(2)$, and $\mathfrak{m}' = \langle A_1, A_2, X_1, X'_1, (X_2)_{\mathfrak{p}}, (X'_2)_{\mathfrak{p}}, U_1, U_2, P_1, P'_1, P_2, P'_2\rangle$. This gives the description $A_{SU(3,2)} \equiv Sl(2, \mathbb{C})(U(1) \times \mathbb{R})N_{\Psi_1}/(U(1) \times SU(2))$. The associated structure S' coincides with the above structure S^λ, for $\lambda = 0$.

The case $\Psi = \Psi_2$. Then $[\Psi_2] = \{\pm 2f_2, \pm f_2\}$ and $\mathfrak{p}_{\Psi_2} = \mathfrak{l}'_{\Psi_2} + \mathfrak{e}'_{\Psi_2} + \mathfrak{a}_{\Psi_2} + \mathfrak{n}_{\Psi_2}$, where $\mathfrak{e}'_{\Psi_2} = i \cdot \mathbb{R}\mathrm{diag}(2, 2, -3, -3, 2) = \langle 3C_1 - 2C_2\rangle$, $\mathfrak{a}_{\Psi_2} = \langle A_1\rangle$, $\mathfrak{n}_{\Psi_2} = \langle X_1, X'_1, X_2, X'_2, U_1, P_1, P'_1\rangle$, and $\mathfrak{l}'_{\Psi_2} = \langle A_2\rangle + \mathfrak{e}'^{\perp}_{\Psi_2} + \langle U_2, V_2, P_2, Q_2, P'_2, Q'_2\rangle$, $\mathfrak{e}'^{\perp}_{\Psi_2} = \langle C_2\rangle \subset Z_{\mathfrak{k}}(\mathfrak{a})$, and we have

$$\mathfrak{l}'_{\Psi_2} = \left\{ \begin{pmatrix} i(r+s) & v & 0 & 0 & w \\ -\bar{v} & -ir & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \bar{w} & \bar{z} & 0 & 0 & -is \end{pmatrix} :\ r, s \in \mathbb{R},\ v, w, z \in \mathbb{C} \right\} \cong \mathfrak{su}(2, 1).$$

For each connected closed subgroup E of P_{Ψ_2} whose Lie algebra $\mathfrak{e}$ is a nontrivial subspace of $\mathfrak{e}'_{\Psi_2} + \mathfrak{a}_{\Psi_2}$, $\mathfrak{e} \neq \mathfrak{e}'_{\Psi_2}$, we get a cocompact subgroup $\hat{G} = LEN_{\Psi_2}$ of $SU(3, 2)$, with $L \cong SU(2, 1)$, which acts transitively on $A_{SU(3,2)}$.

If $\dim \mathfrak{e} = 1$, we obtain $A_{SU(3,2)} \equiv SU(2, 1)\mathbb{R}N_{\Psi_2}/U(2)$. In fact, for each one-dimensional subspace $\mathfrak{e}$ of $\mathfrak{e}'_{\Psi_2} + \mathfrak{a}_{\Psi_2} = \langle 3C_1 - 2C_2, A_1\rangle$, with $\mathfrak{e} \neq \mathfrak{e}'_{\Psi_2}$, we have a reductive decomposition. We can suppose that $\mathfrak{e} = \mathfrak{e}_\lambda$ is generated by $\lambda(3C_1 - 2C_2) + A_1$, for a certain $\lambda \in \mathbb{R}$. So we get a one-parameter family of reductive decompositions $\hat{\mathfrak{g}}^\lambda = \mathfrak{h} + \mathfrak{m}^\lambda$, where $\mathfrak{h} = \hat{\mathfrak{g}}^\lambda \cap \mathfrak{k} = \mathfrak{l}'_{\Psi_2} \cap \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(2)) = \langle C_2, (U_2)_{\mathfrak{k}}, (P_2)_{\mathfrak{k}}, (P'_2)_{\mathfrak{k}}\rangle \cong \mathfrak{u}(2)$, and $\mathfrak{m}^\lambda = \langle \lambda(3C_1 - 2C_2) + A_1, A_2, X_1, X'_1, X_2, X'_2, U_1, (U_2)_{\mathfrak{p}}, P_1, P'_1, (P_2)_{\mathfrak{p}}, (P'_2)_{\mathfrak{p}}\rangle$. The associated one-parameter family of homogeneous structures $S = S^\lambda$ is given at o by the values in Table 3 except that $S_{A_2}(\cdot) = S_{U_2}(\cdot) = S_{P_2}(\cdot) = S_{P'_2}(\cdot) = 0$ and with $\lambda_1 = 3\lambda$, $\lambda_2 = -2\lambda$. Equations Eq. (1.1) are satisfied with the following nonzero values of θ^i at o: $\theta^1(A_1) = 5\lambda$, $\theta^1(U_1) = 1$, $\theta^2(X_1) = -\theta^2(X_2) = \theta^3(X'_1) = -\theta^3(X'_2) = -2$.

We have $S = \Theta + T$, with $\Theta \in \mathcal{QK}_{12} \setminus \mathcal{QK}_1 \cup \mathcal{QK}_2$. Since moreover we have $\vartheta = \frac{1}{28}\langle 7A_1 + 5\lambda U_1, \cdot\rangle$, $(T - T^\vartheta)_{X_1 U_2 X'_2} = \frac{3}{4}$, and $F(T - T^\vartheta)_{X_1 U_2 X'_2} = -\frac{1}{2}$, the tensor S has a nonzero component in each primitive subspace $\mathcal{QK}_i$, $i, \ldots, 5$.

If $\dim\mathfrak{e} = 2$, then $\mathfrak{e} = \langle 3C_1 - 2C_2, A_1\rangle \cong \mathfrak{u}(1)\oplus\mathbb{R}$, and we have the reductive decomposition $\hat{\mathfrak{g}}' = \mathfrak{h}' + \mathfrak{m}'$, where $\mathfrak{h}' = \hat{\mathfrak{g}}'\cap\mathfrak{k} = (\mathfrak{l}'_{\Psi_2} + \mathfrak{e}'_{\Psi_2})\cap\mathfrak{s}(\mathfrak{u}(3)\oplus\mathfrak{u}(2)) = \langle C_1, C_2, (U_2)_{\mathfrak{k}}, (P_2)_{\mathfrak{k}}, (P'_2)_{\mathfrak{k}}\rangle \cong \mathfrak{u}(1)\oplus\mathfrak{u}(2)$, and $\mathfrak{m}'_{\Psi_2} = \langle A_1, A_2, X_1, X'_1, X_2, X'_2, U_1, (U_2)_{\mathfrak{p}}, P_1, P'_1, (P_2)_{\mathfrak{p}}, (P'_2)_{\mathfrak{p}}\rangle$, which is associated with the description $A_{SU(3,2)} \equiv SU(2,1)(U(1)\times\mathbb{R})N_{\Psi_2}/(U(1)\times U(2))$. The corresponding structure S' coincides with the above structure S^{λ}, for $\lambda = 0$.

2.3. The quaternionic hyperbolic space $A_{Sp(3,1)} = \mathbb{H}\mathrm{H}(3)$

The Lie algebra $\mathfrak{sp}(3,1)$ can be described as the subalgebra of $\mathfrak{gl}(8,\mathbb{C})$ of all matrices of the form

$$X = \begin{pmatrix} Z & P^{\mathrm{T}} & W & Q^{\mathrm{T}} \\ \bar{P} & ic & Q & \alpha \\ -\bar{W} & \bar{Q}^{\mathrm{T}} & \bar{Z} & -\bar{P}^{\mathrm{T}} \\ \bar{Q} & -\bar{\alpha} & -P & -ic \end{pmatrix}, \tag{2.6}$$

where $Z = \begin{pmatrix} ia_1 & z_2 & z_3 \\ -\bar{z}_2 & ia_2 & z_1 \\ -\bar{z}_3 & -\bar{z}_1 & ia_3 \end{pmatrix} \in \mathfrak{u}(3)$, $W = \begin{pmatrix} u_1 & w_2 & w_3 \\ w_2 & u_2 & w_1 \\ w_3 & w_1 & u_3 \end{pmatrix}$is complex symmetric, $c\in\mathbb{R}$, $\alpha\in\mathbb{C}$, and $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3) \in \mathbb{C}^3$. The involution τ of $\mathfrak{sp}(3,1)$ given by $\tau(X) = -\bar{X}^{\mathrm{T}}$ defines the Cartan decomposition $\mathfrak{sp}(3,1) = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \left\{\begin{pmatrix} Z & 0 & W & 0 \\ 0 & ic & 0 & \alpha \\ -\bar{W} & 0 & \bar{Z} & 0 \\ 0 & -\bar{\alpha} & 0 & -ic \end{pmatrix}\right\} \cong \mathfrak{sp}(3)\oplus\mathfrak{sp}(1), \qquad \mathfrak{p} = \left\{\begin{pmatrix} 0 & P^{\mathrm{T}} & 0 & Q^{\mathrm{T}} \\ \bar{P} & 0 & Q & 0 \\ 0 & \bar{Q}^{\mathrm{T}} & 0 & -\bar{P}^{\mathrm{T}} \\ \bar{Q} & 0 & -P & 0 \end{pmatrix}\right\}.$$

The element A_0 of $\mathfrak{p}$ obtained by taking $P = (1,0,0)$ and $Q = (0,0,0)$ generates a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of $\mathfrak{sp}(3,1)$. The set of roots Σ corresponding to $\mathfrak{a}$ is $\Sigma = \{\pm f_0, \pm 2f_0\}$, where $f_0 \in \mathfrak{a}^*$ is given by $f_0(A_0) = 1$. The set $\Pi = \{f_0\}$ is a system of simple roots and the corresponding positive root system is $\Sigma^+ = \{f_0, 2f_0\}$. We have generators $X_j, Y_j, X'_j, Y'_j, U_j, V_j, U'_j, V'_j$ of the root spaces $\mathfrak{g}_f$, $f\in\Sigma$, which are represented by the matrix X in (2.6) as follows: X_1 (if $p_1 = i, c = -a_1 = 1$), Y_1 (if $p_1 = -i, c = -a_1 = 1$), U_1 (if $q_1 = u_1 = \alpha = 1$), V_1 (if $q_1 = -u_1 = -\alpha = -1$), U'_1 (if $q_1 = u_1 = \alpha = i$), V'_1 (if $q_1 = -u_1 = -\alpha = -i$), X_j (if $p_j = z_j = 1$), Y_j (if $p_j = -z_j = -1$), X'_j (if $p_j = -z_j = i$), Y'_j (if $p_j = z_j = -i$), U_j (if $q_j = w_j = 1$), V_j (if $q_j = -w_j = -1$), U'_j (if $q_j = w_j = i$), V'_j (if $q_j = -w_j = -i$), for $j = 1, 2$, with all other entries zero for each one of the 22 cases. We have $\mathfrak{g}_{2f_0} = \langle X_1, U_1, U'_1\rangle$, $\mathfrak{g}_{f_0} = \langle X_2, X'_2, U_2, U'_2, X_3, X'_3, U_3, U'_3\rangle$, $\mathfrak{g}_{-2f_0} = \langle Y_1, V_1, V'_1\rangle$, $\mathfrak{g}_{-f_0} = \langle Y_2, Y'_2, V_2, V'_2, Y_3, Y'_3, V_3, V'_3\rangle$. We have the Iwasawa decomposition $\mathfrak{sp}(3,1) = \mathfrak{k}+\mathfrak{a}+\mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_{f_0} + \mathfrak{g}_{2f_0} = \langle X_1, U_1, U'_1, X_j, X'_j, U_j, U'_j\rangle_{j=1,2}$. The centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is

$$Z_{\mathfrak{k}}(\mathfrak{a}) = \left\{\begin{pmatrix} ia_1 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & ia_2 & z & 0 & 0 & u_2 & w & 0 \\ 0 & -\bar{z} & ia_3 & 0 & 0 & w & u_3 & 0 \\ 0 & 0 & 0 & ia_1 & 0 & 0 & 0 & -u_1 \\ -\bar{u}_1 & 0 & 0 & 0 & -ia_1 & 0 & 0 & 0 \\ 0 & -\bar{u}_2 & -\bar{w} & 0 & 0 & -ia_2 & z & 0 \\ 0 & -\bar{w} & -\bar{u}_3 & 0 & 0 & -z & -ia_3 & 0 \\ 0 & 0 & 0 & \bar{u}_1 & 0 & 0 & 0 & -ia_1 \end{pmatrix} : \begin{array}{l} a_j\in\mathbb{R}, \\ u_j, w, z\in\mathbb{C} \\ (1\leqslant j\leqslant 3) \end{array}\right\},$$

and $Z_{\mathfrak{k}}(\mathfrak{a}) \cong \mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$. We consider the basis $\{B_l, C_j, D_j, F_j\}_{1\leqslant l\leqslant 4, 1\leqslant j\leqslant 3}$ of $Z_{\mathfrak{k}}(\mathfrak{a})$ whose elements are defined as follows: B_1 (if $z = 1$), B_2 (if $z = i$), B_3 (if $w = 1$), B_4 (if $w = i$), C_1 (if $a_1 = 1$), C_2 (if $u_1 = 1$), C_3 (if $u_1 = i$), D_1 (if $a_2 = 1$), D_2 (if $u_2 = 1$), D_3 (if $u_2 = i$), F_1 (if $a_3 = 1$), F_2 (if $u_3 = 1$), F_3 (if $u_3 = i$), with all other entries zero for each one of the 13 cases. The subspaces $\langle C_1, C_2, C_3\rangle$ and $\langle B_l, D_j, F_j\rangle_{1\le l\le 4, 1\le j\le 3}$ are ideals of $Z_{\mathfrak{k}}(\mathfrak{a})$ isomorphic to $\mathfrak{sp}(1)$ and $\mathfrak{sp}(2)$, respectively. Moreover, $\langle D_1, D_2, D_3\rangle$ and $\langle F_1, F_2, F_3\rangle$ are Lie subalgebras of $Z_{\mathfrak{k}}(\mathfrak{a})$ isomorphic to $\mathfrak{sp}(1)$.

The elements of $\mathfrak{k} = \mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$ defined by the matrices of the form (2.6) given by

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

generate a compact ideal $\mathfrak{u} \cong \mathfrak{sp}(1)$ of $\mathfrak{k}$, and the isotropy representation $\mathfrak{u} \to \mathfrak{gl}(\mathfrak{p})$ defines a quaternionic Kähler structure on $A_{Sp(3,1)}$. From the isomorphisms $\mathfrak{p} \cong \mathfrak{sp}(3,1)/\mathfrak{k} \cong \mathfrak{a}+\mathfrak{n}$ we obtain the complex structures J_i $(i = 1, 2, 3)$ acting on $\mathfrak{a} + \mathfrak{n}$, which are given in the following table.

	A_0	X_1	U_1	U_1'	X_2	X_2'	U_2	U_2'	X_3	X_3'	U_3	U_3'
J_1	$-X_1$	A_0	U_1'	$-U_1$	$-X_2'$	X_2	U_2'	$-U_2$	$-X_3'$	X_3	U_3'	$-U_3$
J_2	$-U_1$	$-U_1'$	A_0	X_1	$-U_2$	$-U_2'$	X_2	X_2'	$-U_3$	$-U_3'$	X_3	X_3'
J_3	$-U_1'$	U_1	$-X_1$	A_0	$-U_2'$	U_2	$-X_2'$	X_2	$-U_3'$	U_3	$-X_3'$	X_3

The basis $\{A_0, X_1, U_1, U_1', X_j, X_j', U_j, U_j'\}_{j=2,3}$ of $\mathfrak{a} + \mathfrak{n}$ is orthonormal with respect to the scalar product $\langle , \rangle$ defined in $\mathfrak{a} + \mathfrak{n}$ by the isomorphism $\mathfrak{p} \cong \mathfrak{a} + \mathfrak{n}$ and $\frac{1}{40} B_{|\mathfrak{p}\times\mathfrak{p}}$, where B is the Killing form of $\mathfrak{sp}(3, 1)$, and $(\mathfrak{a} + \mathfrak{n}, \langle , \rangle, J_1, J_2, J_3)$ is a quaternion–Hermitian vector space.

2.3.1. Homogeneous descriptions of $A_{Sp(3,1)}$ and homogeneous quaternionic Kähler structures

We will now obtain the homogeneous descriptions of $\mathbb{H}\mathrm{H}(3)$ and the corresponding homogeneous quaternionic Kähler structures. There are only two parabolic subalgebras of $\mathfrak{sp}(3, 1)$ and they are parametrized by the subsets Π and $\emptyset$ of $\Pi = \{f_0\}$.

The case $\Psi = \Pi$. In this case, $\mathfrak{e}'_{\Pi} = \mathfrak{a}_{\Pi} = \mathfrak{n}_{\Pi} = \{0\}$, and hence the refined Langlands decomposition is $\mathfrak{p}_{\Pi} = \mathfrak{sp}(3, 1) + \{0\} + \{0\} + \{0\}$. By Theorem 3, the only transitive action coming from $\Psi = \Pi$ is that of the full isometry group $Sp(3, 1)$. This gives the description of $A_{Sp(3,1)} = \mathbb{H}\mathrm{H}(3)$ as the symmetric space $Sp(3, 1)/(Sp(3)\times Sp(1))$. The associated reductive decomposition is the Cartan decomposition $\mathfrak{sp}(3, 1) = (\mathfrak{sp}(3)\oplus\mathfrak{sp}(1))+\mathfrak{p}$, and the corresponding homogeneous quaternionic Kähler structure is $S = 0$.

The case $\Psi = \emptyset$. We have $\mathfrak{l}' = \mathfrak{a} + Z_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{l}'_{\emptyset} + \mathfrak{e}'_{\emptyset} + \mathfrak{a}_{\emptyset}$, with $\mathfrak{l}'_{\emptyset} = \{0\}$, $\mathfrak{e}'_{\emptyset} = Z_{\mathfrak{k}}(\mathfrak{a})$, and $\mathfrak{a}_{\emptyset} = \mathfrak{a}$. The refined Langlands decomposition of the corresponding parabolic subalgebra is $\mathfrak{p}_{\emptyset} = \{0\} + Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} + \mathfrak{n} = \{0\} + (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)) + \mathfrak{a} + (\mathfrak{g}_{f_0} + \mathfrak{g}_{2f_0})$. For each connected closed subgroup E of $E'_{\emptyset}A \cong Sp(2)Sp(1)\mathbb{R}$ we get a cocompact subgroup EN of $Sp(3, 1)$. By Theorem 4, in order to get a transitive action on $A_{Sp(3,1)}$ it is sufficient that the projection $\mathfrak{e} \subset Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} \to \mathfrak{a}$ be surjective.

Suppose that $\mathfrak{e}$ is an one-dimensional subspace of $Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a} = \langle A_0, B_k, C_l, D_l, F_l\rangle_{1\le k\le 4, 1\le l\le 3}$ such that the projection of $\mathfrak{e}$ to $\mathfrak{a}$ is an isomorphism. Then the Lie subalgebra $\mathfrak{e} + \mathfrak{n}$ of $\mathfrak{sp}(3, 1)$ generates a connected Lie subgroup $\hat{G} = EN$ which acts simply transitively on $A_{Sp(3,1)}$. We can suppose that $\mathfrak{e}$ is generated by one element of the form $\hat{A}_0 = A_0 + \sum_{k=1}^{4} \gamma_k B_k + \sum_{l=1}^{3} (\lambda_l C_l + \mu_l D_l + \nu_l F_l)$, where $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \mathbb{R}^4$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$, $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$. This defines the reductive decomposition $\hat{\mathfrak{g}}^{\lambda,\mu,\nu,\gamma} = \{0\} + \hat{\mathfrak{g}}^{\lambda,\mu,\nu,\gamma}$ associated with the description $A_{Sp(3,1)} = E_{\lambda,\mu,\nu,\gamma}N$, where $\hat{\mathfrak{g}}^{\lambda,\mu,\nu,\gamma} = \langle \hat{A}_0, X_1, U_1, U_1', X_j, X_j', U_j, U_j'\rangle_{j=1,2}$. If all the parameters $\gamma_k, \lambda_l, \mu_l, \nu_l$ are zero, we have $\mathfrak{e} = \mathfrak{a}$, which gives the usual description of $A_{Sp(3,1)}$ as the solvable Lie group AN. So, we get a 13-parameter family of structures $S^{\lambda,\mu,\nu,\gamma}$ associated with these reductive decompositions. With the identifications in (1.2), we give the values at o of the structure $S = S^{\lambda,\mu,\nu,\gamma}$ corresponding to this reductive decomposition. First, for $S_{A_0}(.)$ we have

$$S_{A_0}A_0 = 0, \qquad S_{A_0}X_1 = 2\lambda_2 U_1' - 2\lambda_3 U_1,$$
$$S_{A_0}U_1 = 2\lambda_1 U_1' + 2\lambda_3 X_1, \qquad S_{A_0}U_1' = -2\lambda_1 U_1 - 2\lambda_2 X_1,$$
$$S_{A_0}X_2 = (\mu_1 - \lambda_1)X_2' + (\lambda_2 - \mu_2)U_2 + (\lambda_3 - \mu_3)U_2' - \gamma_1 X_3 + \gamma_2 X_3' - \gamma_3 U_3 - \gamma_4 U_3',$$
$$S_{A_0}X_2' = (\lambda_1 - \mu_1)X_2 + (\lambda_2 + \mu_2)U_2' - (\lambda_3 + \mu_3)U_2 - \gamma_1 X_3' - \gamma_2 X_3 + \gamma_3 U_3' - \gamma_4 U_3,$$
$$S_{A_0}U_2 = (\lambda_1 + \mu_1)U_2' + (\mu_2 - \lambda_2)X_2 + (\lambda_3 + \mu_3)X_2' - \gamma_1 U_3 + \gamma_2 U_3' + \gamma_3 X_3 + \gamma_4 X_3',$$
$$S_{A_0}U_2' = -(\lambda_1 + \mu_1)U_2 - (\lambda_2 + \mu_2)X_2' + (\mu_3 - \lambda_3)X_2 - \gamma_1 U_3' - \gamma_2 U_3 - \gamma_3 X_3' + \gamma_4 X_3,$$

Table 4

	A_0	X_1	U_1	U_1'	X_2	X_2'	U_2	U_2'	X_3	X_3'	U_3	U_3'
S_{X_1}	$-2X_1$	$2A_0$	0	0	$-X_2'$	X_2	U_2'	$-U_2$	$-X_3'$	X_3	U_3'	$-U_3$
S_{U_1}	$-2U_1$	0	$2A_0$	0	$-U_2$	$-U_2'$	X_2	X_2'	$-U_3$	$-U_3'$	X_3	X_3'
$S_{U_1'}$	$-2U_1'$	0	0	$2A_0$	$-U_2'$	U_2	$-X_2'$	X_2	$-U_3'$	U_3	$-X_3'$	X_3
S_{X_2}	$-X_2$	$-X_2'$	$-U_2$	$-U_2'$	A_0	X_1	U_1	U_1'	0	0	0	0
$S_{X_2'}$	$-X_2'$	X_2	$-U_2'$	U_2	$-X_1$	A_0	$-U_1'$	U_1	0	0	0	0
S_{U_2}	$-U_2$	U_2'	X_2	$-X_2'$	$-U_1$	U_1'	A_0	$-X_1$	0	0	0	0
$S_{U_2'}$	$-U_2'$	$-U_2$	X_2'	X_2	$-U_1'$	$-U_1$	X_1	A_0	0	0	0	0
S_{X_3}	$-X_3$	$-X_3'$	$-U_3$	$-U_3'$	0	0	0	0	A_0	X_1	U_1	U_1'
$S_{X_3'}$	$-X_3'$	X_3	$-U_3'$	U_3	0	0	0	0	$-X_1$	A_0	$-U_1'$	U_1
S_{U_3}	$-U_3$	U_3'	X_3	$-X_3'$	0	0	0	0	$-U_1$	U_1'	A_0	$-X_1$
$S_{U_3'}$	$-U_3'$	$-U_3$	X_3'	X_3	0	0	0	0	$-U_1'$	$-U_1$	X_1	A_0

$$
\begin{aligned}
S_{A_0}X_3 &= (\nu_1-\lambda_1)X_3' + (\lambda_2-\nu_2)U_3 + (\lambda_3-\nu_3)U_3' + \gamma_1 X_2 + \gamma_2 X_2' - \gamma_3 U_2 - \gamma_4 U_2',\\
S_{A_0}X_3' &= (\lambda_1-\nu_1)X_3 + (\lambda_2+\nu_2)U_3' - (\lambda_3+\nu_3)U_3 + \gamma_1 X_2' - \gamma_2 X_2 + \gamma_3 U_2' - \gamma_4 U_2,\\
S_{A_0}U_3 &= (\lambda_1+\nu_1)U_3' + (\nu_2-\lambda_2)X_3 + (\lambda_3+\nu_3)X_3' + \gamma_1 U_2 + \gamma_2 U_2' + \gamma_3 X_2 + \gamma_4 X_2',\\
S_{A_0}U_3' &= -(\lambda_1+\nu_1)U_3 - (\lambda_2+\nu_2)X_3' + (\nu_3-\lambda_3)X_3 + \gamma_1 U_2' - \gamma_2 U_2 - \gamma_3 X_2' + \gamma_4 X_2.
\end{aligned}
$$

The remaining values are given by Table 4.

Eq. (1.1) is satisfied, with the following nonzero values of θ^i at o: $\theta^1(A_0) = 2\lambda_1, \theta^2(A_0) = -2\lambda_2, \theta^3(A_0) = -2\lambda_3$, $\theta^1(X_1) = \theta^2(U_1) = \theta^3(U_1') = 2$. We have $S = \Theta + T$, with $\Theta \in \mathcal{QK}_{12} \setminus \mathcal{QK}_1 \cup \mathcal{QK}_2$, except for $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In this case $\Theta \in \mathcal{QK}_1$, with corresponding 1-form $\theta = 2\langle A_0, \cdot\rangle$. As for T, we first have that $\vartheta = \frac{1}{14}\langle 11A_0 + \lambda_1 X_1 - \lambda_2 U_1 - \lambda_3 U_1', \cdot\rangle$. To find the $\mathcal{QK}_{45}$-component, suppose first that at least one of the parameters λ_i, μ_i, $i = 1, 2, 3$, is nonzero. Computing, we get for instance for $\gamma_3 \neq 0$ that $(T - T^{\vartheta})_{A_1 X_2 U_3} = -\gamma_3$ and $F(T - T^{\vartheta})_{A_1 X_2 U_3} = 2\gamma_3$. This also happens for the other parameters; hence the tensor $S \in \mathcal{QK}_{1345}$. If all the parameters vanish, then a computation with Maple shows that $S \in \mathcal{QK}_{134}$ (cf. [5]).

If $\dim \mathfrak{e} > 1$ there exist other subgroups E of $E'_{\emptyset}A \cong Sp(2)Sp(1)\mathbb{R}$ such that EN acts transitively on $A_{Sp(3,1)}$. Such groups E are isomorphic to some subgroup of $Sp(2)Sp(1)\mathbb{R}$ of the form $U(1)\mathbb{R}$, $U(1)U(1)\mathbb{R}$, $U(1)U(1)U(1)\mathbb{R}$, $Sp(1)\mathbb{R}$, $Sp(1)U(1)\mathbb{R}$, $Sp(1)U(1)U(1)\mathbb{R}$, $Sp(1)Sp(1)\mathbb{R}$, $Sp(1)Sp(1)Sp(1)\mathbb{R}$, $Sp(2)\mathbb{R}$, or $Sp(2)Sp(1)\mathbb{R}$. However, the natural reductive decompositions defined by their actions do not provide new structures.

2.4. Types of homogeneous quaternionic Kähler structures

For each parabolic subalgebra $\mathfrak{p}_{\Psi}$ of the Lie algebra of the full connected isometry group G of each 12-dimensional Alekseevsky space M, we have seen that the subgroups $\hat{G}$ of G acting transitively on M are of the form $\hat{G} = L'_{\Psi} E N_{\Psi}$, where L'_{Ψ} is noncompact semisimple or trivial and N_{Ψ} is nilpotent. Moreover, E is a connected closed subgroup of $E'_{\Psi} A_{\Psi}$ such that the projection of its Lie algebra $\mathfrak{e} \subset \mathfrak{e}'_{\Psi} + \mathfrak{a}_{\Psi}$ to $\mathfrak{a}_{\Psi}$ is surjective (see Theorem 4). If this projection is an isomorphism we say that E is *minimal*; in this case the Lie group E is simply connected and abelian. In particular, we have obtained

Theorem 5. *Let $G = KAN$ be the Iwasawa decomposition of each of the groups $SO_0(4,3)$, $SU(3,2)$, $Sp(3,1)$. The homogeneous descriptions $L'_{\Psi} E N_{\Psi}/H$ for E minimal, and the corresponding types of homogeneous quaternionic Kähler structures of the three Alekseevsky spaces of dimension* 12 *are given in the following table (where the figure in the fifth column, if any, stands for the number n of parameters of the corresponding n-parametric family of homogeneous quaternionic Kähler structures).*

G/K	Ψ	$L'_{\Psi} E N_{\Psi}/H$	dim E	n	type
$A_{SO_0(4,3)}$	Π	$SO_0(4,3)/(SO(4)\times SO(3))$	0		$\{0\}$
	$\emptyset$	AN	3		$\mathcal{QK}_{12345}$
	Ψ_1	$Sl(3,\mathbb{R})A_{\Psi_1}N_{\Psi_1}/SO(3)$	1		$\mathcal{QK}_{12345}$
	Ψ_2	$(Sl(2,\mathbb{R})\times Sl(2,\mathbb{R}))A_{\Psi_2}N_{\Psi_2}/(SO(2)\times SO(2))$	1		$\mathcal{QK}_{12345}$
	Ψ_3	$SO_0(3,2)A_{\Psi_3}N_{\Psi_3}/(SO(3)\times SO(2))$	1		$\mathcal{QK}_{135}$
	Ψ_j	$Sl(2,\mathbb{R})A_{\Psi_j}N_{\Psi_j}/SO(2)\ (j=4,5,6)$	2		$\mathcal{QK}_{12345}$
$A_{SU(3,2)}$	Π	$SU(3,2)/S(U(3)\times U(2))$	0		$\{0\}$
	$\emptyset$	$E_{\lambda,\mu}N$	2	4	$\mathcal{QK}_{12345}$
	$\emptyset$	$AN=E_{0,0}N$	2		$\mathcal{QK}_{12345}$
	Ψ_1	$Sl(2,\mathbb{C})E_{\lambda}N_{\Psi_1}/SU(2)$	1	1	$\mathcal{QK}_{1345}$
	Ψ_2	$SU(2,1)E_{\lambda}N_{\Psi_2}/U(2)$	1	1	$\mathcal{QK}_{12345}$
$A_{Sp(3,1)}$	Π	$Sp(3,1)/(Sp(3)\times Sp(1))$	0		$\{0\}$
	$\emptyset$	$E_{\lambda,\mu,\nu,\gamma}N$	1	13	$\mathcal{QK}_{12345}$
	$\emptyset$	$E_{0,\mu,\nu,\gamma}N$	1	10	$\mathcal{QK}_{1345}$
	$\emptyset$	$AN=E_{0,0,0,0}N$	1		$\mathcal{QK}_{134}$

□

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